MATH 307: Homework #2

Due on: October 26, 2015

Problem 1 *Jordan Normal Form*

For each of the following values of the matrix A, find an invertible matrix P and a matrix N in Jordan normal form such that $P^{-1}AP = N$.

(a)

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

(c)

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(d)

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

(e)

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

(f)

$$A = \begin{pmatrix} 0 & 0 & 24 \\ 1 & 0 & 2 \\ 0 & 1 & -5 \end{pmatrix}$$

(g)

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{pmatrix}$$

(h)

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

(i)

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 4 & 3 & 0 \\ -6 & -3 & 1 \end{pmatrix}$$

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Problem 2 **Matrix Exponential**

For each of the values of the matrix A in the previous problem, determine the value of $\exp(At)$

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Problem 3 Fundamental Matrix

Find a fundamental matrix for each of the following systems of equations

(a)

$$\begin{aligned} x' &= x + y \\ y' &= x - y \end{aligned}$$

(b)

$$\begin{aligned} x' &= -x - 4y \\ y' &= x - y \end{aligned}$$

(c)

$$\begin{aligned} x' &= x + y \\ y' &= 4x - 2y \end{aligned}$$

(d)

$$\begin{aligned} x' &= -x - 4y \\ y' &= x - y \end{aligned}$$

(e)

$$\begin{aligned} x' &= x - y \\ y' &= 5x - 3y \end{aligned}$$

(f)

$$\begin{aligned} x' &= 3x - 4y \\ y' &= x - y \end{aligned}$$

(g)

$$\begin{aligned} x' &= 4x - 8y \\ y' &= 8x - 4y \end{aligned}$$

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Problem 4 Matrix Sine and Cosine

Let A be an $n \times n$ matrix. This problem concerns the matrix valued functions $\sin(At)$ and $\cos(At)$.

(a) Show that $\frac{d}{dt}\sin(At) = A\cos(At)$

(b) Show that $\frac{d}{dt}\cos(At) = -A\sin(At)$

(c) Let $\vec{v}, \vec{w} \in \mathbb{R}^n$. Show that

$$\vec{y}(t) := \cos(At) \cdot \vec{v} + \sin(At)\vec{w}$$

is a solution to the differential equation

$$\vec{y}''(t) = -A^2\vec{y}(t).$$

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Problem 5 Second-order differential equations

Consider the differential equation

$$y''(t) + by'(t) + cy(t) = 0. \tag{1}$$

If we make the substitution, $z(t) = y'(t)$, then we may rewrite Equation (1) as a system of two first-order equations

$$\begin{cases} y'(t) &= z(t) \\ z'(t) &= -cy(t) - bz(t) \end{cases} \tag{2}$$

(a) Show that the characteristic polynomial of Equation (1) is the same as the characteristic polynomial of the matrix associated with the linear system in Equation (2).

(b) Find the fundamental matrix of the system in Equation (2) when $b = 5$ and $c = 4$

(c) Find the fundamental matrix of the system in Equation (2) when $b = 2$ and $c = 5$

(d) Find the fundamental matrix of the system in Equation (2) when $b = 2$ and $c = 1$.

(e) For (b)-(d), explain how the fundamental matrix you found corresponds to the general solution of Equation (1).

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Problem 6 A Zombie Outbreak

A zombie outbreak occurs in the isolated country Fictionland. Assume that the human per-capita birth rate in Fictionland is 0.013 and the per-capita death rate of humans is 0.008. The zombie outbreak leads to the conversion of humans to zombies at a rate of $0.003z(t)$, where $z(t)$ is the zombie population of Fictionland at time t. Humans also destroy the zombies at a rate of $dh(t)$, where $h(t)$ is the population of humans in Fictionland at time t. Assuming that at time $t = 0$, there is an equal population of humans and zombies. For which values of d does the human population eventually die out?

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Problem 7 Uniqueness of Fundamental Matrix

Let $A(t)$ be a matrix continuous on the interval (α, β). Show that if $\Psi(t)$ and $\Phi(t)$ are two fundamental matrices for the equation

$$\vec{y}'(t) = A(t)\vec{y}(t)$$

on the interval (α, β), then there exists a (constant) invertible matrix P so that $\Phi(t) = \Psi(t)P$.

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Problem 8 Nonhomogeneous Equations

For each of the following, find the general solution.

(a)

$$\begin{cases} x' &= 2x - y + e^t \\ y' &= 3x - 2y + t \end{cases}$$

(b)

$$\begin{cases} x' &= x + y + e^{-2t} \\ y' &= 4x - 2y - 2e^t \end{cases}$$

(c)

$$\begin{cases} x' &= 2x - 5y - \cos(t) \\ y' &= x - 2y + \sin(t) \end{cases}$$

(d)

$$\begin{cases} x' &= -4x + 2y + t^3 \\ y' &= 2x - y - t^{-2} \end{cases}$$

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