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Independent Project

**ABSTRACT**

Topology is a branch of Mathematics that is heavily involved with geometric shapes, transformations, and continuity. It wasn’t until the 20th century that Topology became a popular topic to focus in mathematics. This paper will focus on what exactly defines topology, what makes two geometric objects topologically equivalent, the definition of topological spaces, and crucial concepts such as connectedness, path-connected spaces, and finally compactness.

**INTRODUCTION**

One of the most fascinating ideas of Topology is the concept that one geometric object can be continuously transformed into another, but yet remain “topologically equivalent.” To distinguish whether two objects are topologically equivalent requires a concept called connectedness. Since we are dealing with geometric objects, it naturally follows to define what exactly is a topological space? After understanding the basic definition and idea of topology, we will dive straight into a more rigor definition of what it means for two objects to be topologically equivalent. That will depend on another concept we will introduce called connectedness. We begin most examples with sets, and then move up to working with topological spaces. That is why it’s natural to then cover path-connected spaces. After laying down a foundation of concepts that build from one another, we will cover compactness because it paves the way for more interesting material that could be further researched.

**BACKGROUND**

**Topologically Equivalent**

While topology can be defined in a diverse amount of ways, a basic notion is that topology is qualitative geometry. For example, the belief is that two objects can have different shapes but still be considered topologically equivalent. A popular way to visualize a way two objects are different but “topologically” the same, would be to consider a rubber band. A rubber band can be stretched into the shape of a circle or a square, which produces two different shapes, but would still be topologically equivalent. As it turns out, in topology, the shapes of our objects are not necessarily important. If we pick out any point out from a circle, we are left with one arc. If we pick out any point away from a square, we are still left with one arc. Therefore a circle and square are topologically equivalent. A figure eight and circle are not topologically equivalent because a figure eight is made of two circles that are tangent to each other. If we take that point of tangency away, we have two separate arcs. Therefore that doesn’t match the one arc produced by a circle by taking a point from it. Anytime two objects are topologically equivalent, they are defined as *homeomorphic.* Thus from our previous example a square and a circle is homeomorphic. On the other hand, a figure eight and a circle is not homeomorphic. But in reality it can be surprisingly difficult to determine if two geometric objects are homeomorphic because our two object’s shape might be extremely intricate. Therefore before we present a proof regarding homeomorphism, we will define exactly what it means for an object to a topological space.

**Topological Spaces**

Dr. Hatcher states that the definitions of topological spaces are the geometric objects for which continuous functions can be defined. Assuming students have already taken a basic understanding of set theory, we provide a more rigorous definition for topological spaces using open sets.

Definition: A *topological space* is a set X together with a collection O of subsets of X, called *open sets,* such that:

1. The union of any collection of sets in O is in O.
2. The intersection of any finite collection of sets in O is in O.
3. Both the empy set and X are in O.

One might ponder why for condition 2 we are restricted to a finite collection? Why not an infinite collection? Observe that the intersection of all the open intervals (-1/n, 1/n) for n=1,2,... is the single point {zero}. But the single point zero is not open; therefore condition 2 has to be restricted to a finite collection. Now that we have introduced the definition of topological spaces, we can use that to present another definition regarding continuity.

Definition: Let X and Y be topological spaces. A function f: X → Y is continuous if f^(−1)(V) is open in X for each open set V in Y . We call continuous functions maps (A map f is a homeomorphism if f is one–to-one and onto) and its inverse function is also continuous.

If we let X and Y be finite spaces, we can present an interesting lemma, along with its proof.

Lemma 1: A map f: X → X is a homeomorphism if and only if f is either one to one or onto.

*Proof*: By finiteness, one–to–one and onto are equivalent. So we will assume that they hold. Then f induces a bijection 2^(f) from the set 2^(X) of subsets of f to itself. Since f is continuous, if f(O) is open, then so is O. Therefore the bijection 2^(f) must restrict to a bijection from the topology O to itself. GED

This lemma goes to show how easy finite collections are in topology, generally under the field of discrete topology. In a way, this lemma reveals to us the simplicity of discrete topology. That’s why it is more relevant for us focus on sets of infinite collections. So now that we have presented a geometric understanding of topologically equivalent and a more mathematical definition of homeomorphism, let us begin to dissect the concept of connectedness.

**Connectedness**

Sometimes to understand what it means to be connected, giving a counterexample of what is not connected is generally helpful. For example A= [0,1] and B= [2,3] will be disconnected because they are disjoint. Now let us give a formal definition of what is means to be connected.

Definition: ‘A space *X* is *connected* if it cannot be decomposed as the union of two disjoint nonempty open sets. This is equivalent to saying *X* cannot be decomposed as the union of two disjoint nonempty closed sets.’

Another way to define connectedness that might be a little easier to follow is:

Let (X, T) be a topological space. Then it connected provided that the only subsets of X which are both open and closed are the empty set and the whole space X.

Example: Given R, the topology having as basis all the intervals [a,b), is it connected or not? As it turns out we can conclude that it is not connected, because the intervals [a,b) are closed and open since their complements (−∞,a)∪(b,∞) are open. Recall that for a set or object to be connected it cannot be decomposed as the union of two disjoint nonempty closed sets. Therefore we can conclude that [a,b) is not connected.

Theorem 1: An interval [a, b] in R is connected.

While this might seem trivial at this point, the actual proof is quite rigorous. So we will provide Dr. Allen Hatcher’s proof of the theorem:

*Proof:* ‘We may assume that a < bsince it is obvious that [a,a]is connected. Suppose [a, b]is decomposed as the disjoint union of sets Aand Bthat are open in [a, b] with the subspace topology, so they are also closed in[a, b]*.* After possibly changing notation we may assume that ais in A. Since Ais open in [a, b]there is an interval [a,a +epsilon ) contained in Afor some epsilon >0, and hence there is an interval [a, c] ⊂ A, with a < c. The set *C* = { c | [a, c] ⊂ A} is bounded above by b, so it has a least upper bound *L ≤* b. (A fundamental property of R is that any set that is bounded above has a least upper bound.) We know that *L >* aby the earlier observation that there is an interval [a, c] ⊂ Awith c > a. Since no number smaller than Lis an upper bound for *C*, there exist intervals [a, c] ⊂ Awith c ≤ Land carbitrarily close to L. These numbers cthatare in A, hence Lmust also be in Asince Ais closed. Thus we have [a, L] ⊂ A. Now if we assume that L < bwe can derive a contradiction in the following way. Since Ais open and contains [a, L], it follows that Acontains [a, L + epsilon]for some epsilon >0. But this means that *C* contains numbers bigger than L, contradicting the fact that Lwas an upper bound for *C*. Thus the assumption L < b leads to a contradiction, so we must conclude that L = bsince we know L ≤ b. We already saw that [a, L] ⊂ A, so now we have [a, b] ⊂ A. This means that Bmust be empty, and we have shown that it is impossible to decompose [a, b]into two disjoint nonempty open sets. Hence [a, b]is connected.’ (19).

So far we have been dealing with connectedness regarding sets, so it is only natural now to expand out to actual topological spaces!

**Path-Connected Spaces**

Since we have brief understanding of connectedness under our belt with sets, let begin by providing a quick definition of path-connectedness for topological spaces.

Definition: A space X is path-connected if for each pair of points a,b ∈ X there exists a path in X from a to b, that is, a continuous map f: [0,1] 🡪 X with f(0)=a and f(1)=b. A more simplified way of defining path connectedness is if any two points can be connected in it by a path then that topological space is path-connected.

A natural question that students might ponder is if we have a space that is path-connected, is it also connected? We will answer that question by providing this proposition.

Proposition 1: If a space is path-connected, then it is connected.

*Proof:* We will argue by way of contradiction. So suppose X is the disjoint union of nonempty open sets A and B. Choose a point a ∈ A and b ∈ B. If X is path-connected there is a path f: [0,1]→ X from a to b. Then f inverse of A and f inverse of B is nonempty disjoint open sets in [0,1] whose union is all of the [0,1], contradicting that [0,1] is connected. (We know [0,1] is connected by theorem 1).

Having an understanding of path connectedness and connectedness, we can now construct a more useful proposition that will help determine whether two objects are connected and path connected.

Proposition 2: Suppose f: X→Y is continuous and onto. Then:

(a) If X is connected, so is Y*.*

(b) If X is path-connected, so isY*.*

*Proof of a):* Suppose Y is the union of disjoint non empty open sets A and B. Then X is the union of the disjoint open sets f inverse of A and f inverse of B, which are both nonempty if f is into!

*Proof of b):* For any points a, b∈ Ythere exist points a′, b′ ∈ Xwith f (a′) = aand f (b′) = bsince *f* is onto. If *X* is path-connected there is a path *g* : *[*0*,* 1*]*→*X* from a′ to b′. Then the composition f(g) : [0, 1]→Yis a path in Yfrom ato b, so Yis path connected.

This is useful because if two spaces are homeomorphic and one is connected, then so is the other. The argument is analogous to determining if two spaces are path connected. But we actually use this proposition for showing when two spaces are not homeomorphic, if one is path-connected or connected and the other is not.

**Compactness**

The third concept we will cover is the notion of Compactness. Compactness is about a space’s finite property. For instance, spaces which are infinitely large such as R would not be labeled compact. However, we only want compactness to depend on the topology of a space, so it will have to be defined purely in terms of open sets. This means that any space that is homeomorphic to a noncompact space will also be noncompact. Basically what this means is that finite intervals, (a,b) and (a,b] are noncompact even though they are finite. So it naturally leads to the conclusion that the interval [a,b] will be compact because they cannot be stretched to be infinitely large. Before we give the definition of compact, let’s provide some useful definitions:

A **cover** of a set *X* is a collection of sets whose union contains X as a subset.

Let C be a cover of a topological space X. A **subcover** of *C* is a subset of *C* that still covers X.

Then it follows that if we say that C is an **open cover** if each of its members is an open set.

Definition: A topological space X is compact if each open cover of X contains a finite part that also covers X. In other words, a space X is compact if every open cover of X contains a finite subcovering.

With the definition given, we can expand on the reason for why R is not compact. It’s because the cover by the open intervals (-n,n) for n=1..,2…3,… has no finite subcover, since it takes infinitely many subcovers to cover all of R.

Let’s consider whether or not the interval (0,1) is compact? It fails to be compact since the cover by the open intervals (1/n,1) for n≥1 has no finite subcover. So another natural question to ask is whether the closed interval [a,b] is compact? We know from the earlier theorem it is connected. As it turns out it is compact! But we will focus on another way a space is compact.

Theorem 2: If f: X→Y is continuous and onto, and if X is compact, then so is Y.

*Proof*: Let a cover of Y by open sets O∞ be given. Then the sets f ^(−1)(O∞) form an open cover of X. If X is compact, this cover has a finite subcover. Call this finite subcover f ^(−1)(O1), ··· , f^( −1)(On). Assuming that f is onto, the corresponding sets O1, ··· ,On then cover Y since for each y ∈ Y there exists x ∈ X with f (x) = y, and this x will be in some set f ^(−1)(Oi) of the finite cover of X, so y will be in the corresponding set Oi.

The next two theorems will be provided without the proofs because they will be used to prove theorem 5.

Theorem 3: If [a,b] is a closed interval it is compact.

Theorem 4: If X and Y are compact then so is their product X × Y.

Theorem 5: A subspace X ⊂ R^(n) is compact if and only if it is closed and bounded.

*Proof*: First let’s assume X is bounded and closed.

If we assume X is bounded, then it lies in a ball of finite radius and hence in some closed cube [−r, r] × ··· × [−r, r]. This cube is compact by theorem 4. Since X is a closed subset of a compact space, it is also compact by Theorem 3. Now for the converse, suppose X is compact.

Now we will show the other direction, but will do it by proof of contrapositive.

The collection of all open balls in R^(n) centered at the origin and of arbitrary radius forms an open cover of X, so there is a finite subcover, which means X is contained in a single ball of finite radius, the largest radius of the finitely many balls covering X. Hence X is bounded. To show X is closed if it is compact, suppose x is a limit point of X that is not in X. Then every neighborhood of x contains points of X. In particular each open ball Br (x) of radius r centered at x contains points of X, so the same is true also for the closed balls Br (x). The complements R^(n)−Br (x) form an open cover of X as r varies over (0,∞) since their union is R^(n) − {x} and x ∉ X. This open cover of X has no finite subcover since each Br contains points of X. Thus we have shown that if X is not closed, it is not compact.

**CONCLUSION**

Topology is an interesting but yet challenging field of mathematics. While students may have had exposure to continuity, open and closed sets, and geometric observations, even the basic concept of topology is abstract. Just the idea that one geometric object can be transformed into another shape but still be considered the same is difficult to understand. But after we introduced a more precise meaning of what it meant to be a topological space, we learn that the shapes of our objects are not important, the properties are. One property we studied was connectedness, and how closely related it was to deciphering whether or not two shapes are topologically equivalent. It was my goal early on to provide definitions, proofs, and lemmas about sets before I moved on to topological spaces to ease the learning curve. That is where the concept of path-connected was introduced. We proved that is if a space is path-connected, then it is also connected. So it became natural for students to see how previous ideas and concepts laid the foundation for more rigorous theorems and definitions. The final concept covered was compactness. Now, compactness is a very interesting topic because during the early stages of topology, there was no obvious way to generalize what closed and bounded was. From the definition and theorems of compactness, it really helped shape our understanding of what it means to be closed and bounded. It also follows from compactness a lot of new interesting material that students can further explore (See future works.)

**FUTURE WORKS**

Separation axioms are ways to consider natural restrictions making a structure closer to being metrizable (metric d on X is sometimes referred to as a distance function on X). A lot of these restricted spaces are denoted a certain level, for example, The Hausdorff Axiom is labeled T2.

Definition: To say that a topological space (X,T) is metrizable if there is a metric d on X so that T = Td. (we really do not use any definitions of metric for the two spaces I will consider, but it will be useful to know for different spaces I present)

Definition: A Hausdorff axiom is any two distinct points possess disjoint neighborhoods.

Because separation axioms are popular we can get pictures to help students understand conceptually what is going on. This one is courtesy of Wikipedia on how a Hausdorff space looks like:



Proposition 3: A compact subspace of a Hausdorff space is closed.

*Proof*: Let X be Hausdorff with A ⊂ X compact. The main work will be to show that for each x ∈ X − A there exist disjoint open sets U and V in X such that x ∈ U and A ⊂ V. This will imply that A is closed since no point x in the complement of A can be a limit point of A since it has a neighborhood U disjoint from A. Now we construct U and V with the desired properties. The point x ∈ X−A and an arbitrary point a ∈ A have disjoint open neighborhoods Ux and Va in X since X is Hausdorff. As a ranges over A the open sets Va form an open cover of A. Since A is compact, this open cover has a finite subcover. Call this subcover V1, ··· ,Vn , and let the sets Ux that correspond to these Vi ’s be labeled U1, ··· ,Un. The intersection U = U1 ∩ ··· ∩ Un is an open neighborhood of x that is disjoint from each Vi and hence also from the union V = V1 ∪···∪Vn, which is an open set containing A.

We provide a corollary to display how compact and homeomorphism is related to a Hausdorff space.

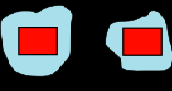
Corollary: A map f: X→Y from a compact space X to a Hausdorff space Y is a homeomorphism if it is continuous, one-to-one, and onto.

*Proof:* All that needs to be checked is that f −1 is continuous, and to do this it suffices to show that if C ⊂ X is closed in X then f (C) is closed in Y. But since C is a closed subset of a compact space, it is compact; hence f (C) is compact and therefore also closed since Y is Hausdorff.

Now let us consider another space called the normal space. It is similar to the Hausdorff space, but it has a stronger restriction.

Definition: a *Normal Space* is if A and B are disjoint closed sets in X, then there are disjoint open sets U and V in X with A ⊂ U and B ⊂ V.

We prove another picture of a normal space, which was again extracted from Wikipedia:



As you might imagine, it is very similar to the Hausdorff space, so it begs the question if the Hausdorff space is related to the normal space? And in fact it is! As I mentioned, basic concepts we covered in our paper will come up quite frequently with more interesting spaces.

Proposition 4: A compact Hausdorff space is normal.

*Proof:* Let A and B be disjoint closed sets in a compact Hausdorff space X. In particular this means that A and B are compact since they are closed subsets of a compact space. By the argument in the proof of proposition 3 we know that for each x ∈ A there exist disjoint open sets Ux and Vx with x ∈ Ux and B ⊂ Vx. Letting x vary over A, we have an open cover of A by the sets Ux, so there is a finite subcover. Let U be the union of the sets Ux in this finite subcover and let V be the intersection of the corresponding sets Vx. Then U and V are disjoint open neighborhoods of A and B. Therefore we have proved that a compact Hausdorff satisfies the definition of being a normal space.

I provided two different spaces in our separation axiom body, but there are many more spaces with particular restriction students can explore.

Definition: A *Kolmogrov space* is if any two distinct points in X are topologically distinguishable.

Definition: We say X is a *Frechet* if any two distinct points in X are separated.

Definition: We say X is a *completely* *Hausdorff* if any two distinct points in X are separated by a continuous function.

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