Noncommutative Crepant Resolutions

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1 Introduction

1.1 Crepant Resolutions of Singularities

Definition 1. Let X be a scheme over a field k. A **resolution of singularities** of X is a proper, birational map $\pi : Y \to X$ from a smooth scheme Y.

Here by proper, we mean that π is separated of finite type and universally closed, ie. for any $f : Z \to X$, the pullback $Y \times_X Z \to Z$ is a closed map. By birational we mean that there exists a dense open subset $U \subseteq X$ such that $f^{-1}(U)$ is dense in Y and f restricts to an isomorphism of $f^{-1}(U) \cong U$. If X, Y are defined over $\mathbb{C}$, this is equivalent to the condition that the preimage of every analytic compact is compact. Smooth means that the the local defining equations for Y have a jacobian matrix of full rank near every point. If k is perfect, then this is equivalent to the condition that for all $y \in Y$ we have $\dim_{\kappa(y)} m_y/m_y^2 = \dim(\mathcal{O}_{Y,y})$ where here m_y is the maximal ideal of the local ring $\mathcal{O}_{Y,y}$ with residue field $\kappa(y)$.

These conditions are all more naturally expressed when X and Y are defined over $\mathbb{C}$, in which case there are analytic versions X_{an}, Y_{an} and f_{an} of the data. In this case, properness means that the preimage of compact (analytic) subsets of Y are compact and smoothness means that Y_{an} is smooth. Note that the properness condition is *essential*, since otherwise the smooth locus X_{sm} of X would always provide a trivial resolution of singularities $X_{sm} \to X$, and yet not tell us any new information about X. Properness forces Y to not only be smooth but to be a smoothing out of the singular points of X, ie. each singular point p of X will have an analytic compact neighborhood which is mapped onto by a smooth, analytic compact neighborhood of Y.

Example 2. Consider the scheme $X = \text{Spec}(R)$ for $R = \mathbb{C}[w, x, y, z]/(wx - yz))$. The scheme X by definition has a natural embedding $f : X \to \mathbb{A}^4_{\mathbb{C}}$ with Jacobian matrix df of rank 1 everywhere except for $p = (0, 0, 0, 0)$ where X has a singularity.

Let $I = (w, y)$ and $A = \mathbb{C}[w, x, y, z]$. The blow-up of $\mathbb{A}^4_{\mathbb{C}}$ at I is the scheme $\text{Bl}_p(\mathbb{A}^4_{\mathbb{C}}) := \text{Proj}\left(\bigoplus_{n=0}^{\infty} I^n t^n\right) \cong \text{Proj}(A[u, v]/(wv - yu)) \subseteq \mathbb{P}^1_{\mathbb{C}} \times \mathbb{A}^4_{\mathbb{C}}$, and comes equipped with a natural map $\pi : \text{Bl}_p(\mathbb{A}^4_{\mathbb{C}}) \to \mathbb{A}^4_{\mathbb{C}}$. The strict transform Y of X under π gives a resolution of singularities $\pi : Y \to X$ of X. In particular, calculation of the jacobian of the relations of Y shows that Y is smooth.

For sake of complete concreteness, let's describe the set of closed points of Y in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{A}^4_{\mathbb{C}}$. The image of Y is cut out by the relations $vw - uy$, $wx - yz$, and $ux - vz$ in the graded ring $A[u, v]$ (with u, v in degree 1 and w, x, y, z in degree 0), so the closed points of Y are given by

$$Y_0 = \{([a_1 : 1], a_1a_3, a_2, a_3, a_1a_2) : a_i \in \mathbb{C}\} \cup \{([1 : b_1], b_2, b_1b_3, b_1b_2, b_3) : b_i \in \mathbb{C}\}$$

and the projection π of Y_0 to the closed points of X_0 in $\mathbb{A}^4_{\mathbb{C}}$ is simply the projection onto the last four components. Note that the fiber of the singular point p of X under π is a copy of $\mathbb{P}^1_{\mathbb{C}}$.

As the previous example demonstrates, a resolution of singularities Y of an affine scheme X may not be affine, even if X is affine. In fact, a resolution of singularities is almost never affine because of the imposed condition that the map be proper. By blowing up multiple times, one can construct multiple non-isomorphic resolutions of singularities. This leads to the following natural question.

Question 3. Is there a smallest/simples/best resolution of singularities $\pi : Y \to X$?

This is the starting point of the minimal model program: resolutions of the singularities of X are viewed as smooth representatives of the birational equivalence class of X and by considering things like the existence (or lack thereof) of blow-downs one can arrive at natural candidates for minimal models. This works as stated in low dimension over $\mathbb{C}$, but has a more complicated story in higher dimensions as well as possible existence issues in positive characteristic.

One kind of resolution in particular satisfies our intuition of smallest/simplest/best when it exists.

Definition 4. A morphism $f : Y \to X$ is called **crepant** if the natural morphism $f^*\omega_X \to \omega_Y$ is an isomorphism. A crepant morphism which is a resolution of singularities is called a **crepant resolution**.

Crepant resolutions are minimal in the following sense.

Proposition 5. Suppose that $\pi : Y \to X$ is a crepant resolution with Y quasi-projective that factors through a resolution $f : Z \to X$. Then $Y \cong Z$.

Proof. Consider the chain of morphisms $Y \xrightarrow{g} Z \xrightarrow{f} X$ whose composition is π. Since f and π are birational, g is birational. Since Y is also quasi-projective, the exceptional locus E of g is either empty or a union of codimension 1 subvarieties.

To see this, assume E is not a union of codimension 1 subvarieties and is not empty. Then there exists a curve $C \subseteq E$ which contracts to a point $p \in Z$ under g and which is not contained in a codimension 1 irreducible component of E. Furthermore, since Y is quasi-projective it has an ample divisor D which must intersect nontrivially with C. It's image $f_*(D)$ is a divisor on Z which is Cartier since Z is smooth and which intersects with p. It's strict transform $f^*f_*(D)$ does not intersect with C. However, since f_*D

is Cartier $f^*f_*(D) = D$ which is a contradiction. Thus either E is a divisor or or E is empty.

Since π is crepant, the exceptional locus E'' of π is small ie. codimension ≥ 2. Furthermore, since $E \subseteq E''$ this tells us E is small and thus empty by the previous paragraph. Thus g is an isomorphism. $\square$

Crepant resolutions do not exist in general. In two dimensions over $\mathbb{C}$, crepant resolutions of Gorenstein quotient singularities exist and are unique. In three dimensions, crepant resolutions of Gorenstein quotient singularities exist, but we lose uniqueness due to the existence of "flops" created by sequences of blow-ups and blow-downs.

Example 6. Let X be as in Example 2. Let $J = (w, z)$ and $A = \mathbb{C}[w, x, y, z]$. The blow-up of $\mathbb{A}^4_{\mathbb{C}}$ at J is the scheme $\mathrm{Bl}_p(\mathbb{A}^4_{\mathbb{C}}) := \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} J^n t^n\right) \cong \mathrm{Proj}(A[u, v]/(wv - zu)) \subseteq \mathbb{P}^1_{\mathbb{C}} \times \mathbb{A}^4_{\mathbb{C}}$. The strict transform $\widetilde{Y}$ of X under the natural map $\widetilde{\pi} : \mathrm{Bl}_p(\mathbb{A}^4_{\mathbb{C}}) \to \mathbb{A}^4_{\mathbb{C}}$ gives a resolution of singularities $\widetilde{\pi} : \widetilde{Y} \to X$ of X. The schemes Y and $\widetilde{Y}$ are not isomorphic over X.

In higher dimensions, crepant resolutions need not exist at all.

1.2 Categorical Resolutions of Singularities

Setting aside the issue of the existence of singularities, the lack of uniqueness of crepant resolutions poses a problem in our search for a minimal resolution. However, from a certain categorical point of view this problem does not exist at all.

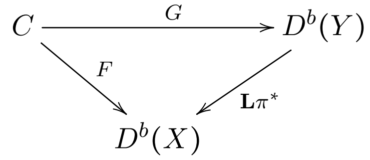
Theorem 7 (Bridgeland[1])**.** Let X be a projective threefold over $\mathbb{C}$ with terminal singularities and let $Y \to X$ and $\widetilde{Y} \to X$ be crepant resolutions of X. Then there is an equivalence of categories $D^b(Y) \cong D^b(\widetilde{Y})$.

Here $D^b(Z)$ denotes the bounded derived category of coherent sheaves on a scheme Z. It is conjectured that this result extends to generalized flops.

Conjecture 8 (Bondal-Orlov)**.** For any generalized flop $Y \dashrightarrow \widetilde{Y}$ between smooth varieties, there is an equivalence of categories $D^b(Y) \cong D^b(\widetilde{Y})$.

This suggests that $D^b(Y)$ may be a natural choice of object encoding the commonality between various crepant resolutions, motivating the following intuitive definition of a categorical resolution of singularities.

Definition 9. A **categorical resolution of singularities** is a category C and a morphism $F : C \to D^b(X)$ such that for some resolution of singularities $\pi : Y \to X$ there is an equivalence of categories $G : C \to D^b(Y)$ making the following diagram commutes



The resolution is called crepant if $Y \to X$ is crepant.

There are other various definitions of categorical resolution of singularities that may be found in the literature. However, our purpose here is to motivate the definition of noncommutative resolutions and for this purpose the above definition is sufficient.

1.3 Noncommutative Crepant Resolutions of Singularties

A later reproving of Bridgelands result came from Van Den Bergh, who proved that both $D^b(Y)$ and $D^b(\widetilde{Y})$ are equivalent because they are equivalent to the bounded, derived category of modules in a certain noncommutative ring $D^b(\Lambda)$.

Example 10. Let X, R, I and J be as in Example 2 and 6, let $\Lambda = \mathrm{End}_R(R \oplus I)$ and $\mathcal{E} = \pi^*(O_X \oplus \widetilde{I})$. Then $\mathbf{R}\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}, -) : D^b(Y) \to D^b(\Lambda)$ is an equivalence of categories. Note also that $J \cong I^{-1}$ and that

$$\Lambda \cong \begin{pmatrix} R & I \\ J & R \end{pmatrix} \cong \mathrm{End}_R(R \oplus J).$$

Then similarly $D^b(\widetilde{Y})$ is equivalent to $D^b(\Lambda)$.

The idea of finding a resolution of a ring R with a noncommutative ring Λ is based on the notion of a categorical resolution of singularities outlined in the previous paragraph.

Definition 11. Let $X = \mathrm{Spec}(R)$. A (categorical) **noncommutative resolution of** R is a ring homomorphism $R \to \Lambda$ inducing a categorical resolution of singularities where the category $C = D^b(\Lambda)$ and $F : C \to D^b(X)$ is $F = \mathbf{L}(- \otimes_R X)$. The noncommutative resolution is crepant if the categorical resolution is.

Theorem 12 (Iyama-Wemyss)**.** Let $R \to \Lambda$ be a noncommutative resolution of singularities. Then Λ is a finitely generated R-module with finite global dimension. Furthermore Λ is crepant if and only if Λ is maximal Cohen-Macaulay and equal to the endomorphism ring of a reflexive R-module.

This theorem motivates the following alternative definition of a noncommutative crepant resolution of singularities.

Definition 13. An (algebraic) **noncommutative crepant resolution of** R is an R-algebra Λ which as an R-module is finitely generated and maximal Cohen-Macaulay, and which is isomorphic to the endomorphism ring of a reflexive R-module.

2 Tilting

Definition 14. Let X be a noetherian scheme. An object $T \in D(X)$ is called a **tilting complex** if T is a perfect complex (isomorphic in $D(X)$ to a bounded complex of finite, locally free sheaves of finite rank), if $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(T,T) = 0$ for $i > 0$, and if the smallest triangulated subcategory containing T and closed under direct summands is $D^{\mathrm{perf}}(X)$. In the case that the complex T has a single component, we call T a **tilting bundle**.

Here $D(X)$ is the derived category of quasi-coherent sheaves on X, while $D^b(X)$ denotes the derived category of coherent sheaves.

The simplest example of a tilting bundle is the bundle $T = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \cdots \oplus \mathcal{O}_X(-n+1)$ which is a tilting bundle for $D^b(\mathbb{P}^n)$. To prove this, we first require Beĭlinsons's resolution of the diagonal

Proposition 15. Let $X = \mathbb{P}^n$ and let $\mathcal{O}_\Delta$ be the structure sheaf of the diagonal Δ of $X \times X$. The exact sequence

$$0 \to \mathcal{O}_X(-n) \boxtimes \Omega^n(n) \to \mathcal{O}_X(-n+1) \boxtimes \Omega^{n-1}(n-1) \to \cdots \to \mathcal{O}_X(1) \boxtimes \Omega^1(1) \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0$$

is a locally free resolution.

Proof. Fix a basis $y_0, \dots, y_n$ of the global sections of $\Omega_X(1)$. Consider the exact sequence on X defined by

$$0 \to \Omega_X(1) \to \mathcal{O}_X^{n+1} \to \mathcal{O}_X(1) \to 0,$$

where the right arrow is defined by sending the degree 0 standard basis $e_0, \dots, e_n$ of the global sections of $\mathcal{O}_X^{n+1}$ to global sections of $\mathcal{O}_X(1)$ by sending $e_i \mapsto y_i$ for all i. Taking the dual of this short exact sequence, using the fact that $\operatorname{Ext}^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) = 0$, we obtain a short exact sequence

$$0 \to \mathcal{O}_X(-1) \to (\mathcal{O}_X^{n+1})^* \to (\Omega_X^*)(-1) \to 0.$$

For each i, let ∂_{y_i} denote the image of the element $\widehat{e_i}$ of the dual basis.

Consider the global section s of $\mathcal{O}(1) \boxtimes (\Omega_X^*)(-1) = (\mathcal{O}(-1) \boxtimes \Omega_X(1))^*$ defined by

$$s = \sum_{i=0}^{n} x_i \boxtimes \partial_{y_i}.$$

The zero locus of s is precisely the diagonal Δ, so s gives rise to an exact sequence

$$\mathcal{O}(-1) \boxtimes \Omega_X(1) \xrightarrow{s} \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0.$$

Taking the Koszul complex of s, we get a chain complex

$$0 \to \wedge^{n+1}(\mathcal{O}(-1) \boxtimes \Omega_X(1)) \to \wedge^n(\mathcal{O}(-1) \boxtimes \Omega_X(1)) \to \cdots \to \mathcal{O}(-1) \boxtimes \Omega_X(1) \xrightarrow{s} \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0.$$

Let $\mathcal{I}$ be the image of $\mathcal{O}(-1) \boxtimes \Omega_X(1)$ in $\mathcal{O}_{X\times X}$. For each $p \in X \times X$, the stalk $\mathcal{O}_{\Delta,p}$ has dimension n, while $\mathcal{O}_{(X\times X),p}$ has dimension $2n$ and $\mathcal{I}_p$ is generated by a sequence of n elements. It follows that the n generators of I_p form a regular sequence in $\mathcal{O}_{(X\times X),p}$. Consequently the Koszul complex corresponding to $(\mathcal{O}(-1) \boxtimes \Omega_X(1))_p \to \mathcal{O}_{(X\times X),p}$ is exact. Thus the above sequence is exact and since for each i we have isomorphisms $\wedge^i(\mathcal{O}(-1) \boxtimes \Omega_X(1)) \cong \mathcal{O}(-i) \boxtimes \Omega_X^i(i)$, we are done. □

We will also require the notion of a Fourier-Mukai transform.

Definition 16. Let X, Y be schemes and $E \in D^b(X \times Y)$. The Fourier-Mukai transform associated to E is the functor

$$\Phi^E_{X \to Y} : D^b(X) \to D^b(Y), \quad F \mapsto \mathbf{R}\pi_{Y*}(\mathbf{L}\pi_X^*(F) \otimes^{\mathbf{L}} E),$$

where here π_X and π_Y are the natural projections.

Note that for any sheaf $\mathcal{F}$ on X we have $\pi_{Y*}(\pi_X^*(\mathcal{F})) = \Gamma(X, \mathcal{F}) \otimes_k \mathcal{O}_Y$. Therefore $\mathbf{R}\pi_{Y*}(\mathbf{L}\pi_X^* = \mathbf{R}\Gamma$ for Γ the functor sending a coherent sheaf $\mathcal{F}$ on X to the sheaf $\Gamma(F) := \Gamma(X, F) \otimes_k \mathcal{O}_Y$ on Y. In particular $\mathbf{R}\Gamma$ sends a complex F to the complex $\Gamma(X, F) \otimes_k \mathcal{O}_Y$ whose differentials are all 0 and whose i'th component has $\dim \Gamma(X, F_i)$ copies of $\mathcal{O}_Y$. When $E = G \boxtimes H$, the projection formula tells us

$$\Phi^{G \boxtimes H}_{X \to Y}(F) = (\mathbf{R}\pi_{Y*}\mathbf{L}\pi_X^*(F \otimes^{\mathbf{L}} G)) \otimes^{\mathbf{L}} H = \mathbf{R}\Gamma(X, -)(F \otimes^{\mathbf{L}} G) \otimes^{\mathbf{L}} H.$$

Moreover, for any $Z \in D^b(X)$ and exact triangle in $D^b(X \times X)$

$$E \to F \to G \to E[1]$$

we obtain an exact triangle in $D^b(Y)$ given by

$$\Phi^E_{X \to Y}(Z) \to \Phi^F_{X \to Y}(Z) \to \Phi^G_{X \to Y}(Z) \to \Phi^E_{X \to Y}(Z)[1].$$

Corollary 17. Let $X = \mathbb{P}^n$. The sheaves $\mathcal{O}_X, \mathcal{O}_X(-1), \dots, \mathcal{O}_X(-n+1)$ generate $D^b(X)$.

Proof. Let $\mathcal{O}_\Delta$ be the structure sheaf of the diagonal Δ of $X \times X$ and let $p_1, p_2 : X \times X \to X$ be the canonical projection maps. For any $E \in D^b(X \times X)$ let $\Phi^E : D^b(X) \to D^b(X)$ be the Fourier-Mukai transform defined by $\Phi^E(Z) = Rp_{1*}(\mathbf{L}p_2^*(Z) \otimes^{\mathbf{L}} E)$. As an important special case, note that by the projection formula

$$\Phi^{\mathcal{O}_\Delta}(Z) = Rp_{1*}(\mathbf{L}p_2^*(Z) \otimes^{\mathbf{L}} \mathcal{O}_\Delta) = Rp_{1*}(\mathcal{O}_\Delta \otimes^{\mathbf{L}} \mathbf{L}p_1^*(Z)) = Rp_{1*}\mathcal{O}_\Delta \otimes^{\mathbf{L}} Z = Z.$$

The resolution of singularities from the previous proposition gives us a series of short exact sequences

$$0 \to \operatorname{img}(s) \to \Omega_{X \times X} \to \Omega_\Delta \to 0.$$

$$0 \to \operatorname{img}(d_2) \to \mathcal{O}(-1) \boxtimes \Omega_X(1) \to \operatorname{img}(s) \to 0$$

and more generally for each i

$$0 \to \operatorname{img}(d_{i+1}) \to \mathcal{O}(-i) \boxtimes \Omega_X^i(i) \to \ker(d_i) \to 0.$$

Therefore for each $Z \in D^b(X)$ we see that $Z = \Phi^{\mathcal{O}_\Delta}(Z)$ is in the triangulated category generated by $\Phi^{\mathcal{O}_{X \times X}}$ and $\Phi^{\mathcal{O}(-i) \boxtimes \Omega_X^i(i)}(Z)$ for $i = 1, \dots, n$. Thus to prove our theorem, it suffices to show that $\Phi^{\mathcal{O}(-i) \boxtimes \Omega_X^i(i)}(Z)$ is in the triangulated category generated by $\mathcal{O}_X, \mathcal{O}_X(-1), \dots, \mathcal{O}_X(-n+1)$.

Via the discussion above, we know

$$\Phi^{\mathcal{O}(-i) \boxtimes \Omega_X^i(i)}(Z) = \mathcal{O}(-i) \otimes^{\mathbf{L}} \Gamma(X, -)(Z \otimes^{\mathbf{L}} \Omega_X^i(i)),$$

which is a shift of the direct sum of $\dim(\Gamma(X, Z \otimes^{\mathbf{L}} \Omega_X^i(i)))$ copies of $\mathcal{O}(-i)$. This completes the proof. $\square$

As a consequence, we have the following theorem.

Theorem 18 (Beǐlinson)**.** The complex

$$T = \mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \cdots \oplus \mathcal{O}_X(-n+1)$$

is a tilting complex on $X = \mathbb{P}^n$.

Proof. Clearly T is a perfect complex and

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(T, T) = \underline{\mathrm{Ext}}_{\mathcal{O}_X}(\mathcal{O}_X, T) \otimes_{\mathcal{O}_X} T^* = 0.$$

for all $i > 0$. By the previous corollary, the direct summands of T generate the category $D^b(X) = D^{\mathrm{perf}}(X)$, so T is a tilting bundle. □

Tilting bundles give rise to categorical equivalences between derived categories of schemes and derived categories of modules on a noncommutative ring Λ.

Theorem 19 (Hille-Van den Bergh)**.** Let X be a scheme, projective over a finite type affine scheme over an algebraically closed field k. Let T be a tilting complex and $\Lambda = \mathrm{End}_{\mathcal{O}_X}(T)$. Then

(a) $\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(T, -)$ induces an equivalence of triangulated categories between $D(X)$ and $D(\Lambda)$ with inverse $- \otimes^{\mathbf{L}}_X T$

(b) if $T \in D^b(X)$ then this restricts to an equivalence of categories between $D^b(X)$ and $D^b(\Lambda)$

(c) if X is smooth, then Λ has finite global dimension

Example 20. Let $X = \mathbb{P}^1_X$ and let $T = \mathcal{O}_X \oplus \mathcal{O}_X(i)$. Then $\mathrm{End}_{\mathcal{O}_X}(\mathcal{O}_X(i), \mathcal{O}_X(j)) = \mathrm{End}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(j-i)) = H^0(X, \mathcal{O}_X(j-i))$ and therefore

$$\Lambda = \begin{pmatrix} k & k \oplus k \\ 0 & k \end{pmatrix} = k\left(\cdot \rightrightarrows \cdot\right),$$

where the rightmost algebra is the path algebra of the quiver.

Bibliography

[1] Tom Bridgeland. Flops and derived categories. *Invent. Math.*, 147(3):613–632, 2002.