Pseudospectral Methods and Instability in Rotating Shallow Water

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W.R. Casper

Department of Mathematics
Louisiana State University

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Outline

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Geophysical Length Scales

- typical length scale of flow in the ocean: $15,000$ km
- typical ocean depth: 3 km
- typical thickness of a sheet of paper: 0.0039 in

On a global scale the ocean is a paper-thin sheet on the surface of a basketball.

- we can model a patch of ocean as a shallow water

The Shallow Water Equations

Shallow Water Equations

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} = \vec{u} \times f\hat{z} - g\nabla\eta$$

$$\eta_t + (H + \eta)\nabla \cdot \vec{u} + \vec{u} \cdot (\nabla\eta) = 0$$

- η free surface height
- H mean depth
- $\vec{u}$ the (horizontal) fluid velocity

(!!) Assumes $H/L \ll 1$ where L is length scale of interest

Non-dimensional Form

Non-dimensional Shallow Water Equations

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} = \frac{1}{\text{Ro}} \vec{u} \times \hat{z} - \frac{1}{\text{Ro}} \nabla \eta$$

$$\eta_t + \left(\frac{\text{Bu}}{\text{Ro}} + \eta \right) \nabla \cdot \vec{u} + \vec{u} \cdot (\nabla \eta) = 0$$

- $\text{Ro} = U/(fL)$
- $\text{Fr} = U/\sqrt{gH}$
- $\text{Bu} = (\text{Ro}/\text{Fr})^2 = (L_d/L)^2$ for L_d Rossby deformation radius

Outline

Linearized Shallow Water

Linearizing around $\eta = 0$ and $\vec{u} = 0$:

$$u = \widetilde{u}(t)e^{ik_x x + ik_y y}, \quad v = \widetilde{v}(t)e^{ik_x x + ik_y y}, \quad \eta = \widetilde{\eta}(t)e^{ik_x x + ik_y y},$$

under which the linearized shallow water equations become

$$\begin{bmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{\eta}/\sqrt{\mathrm{Bu}} \end{bmatrix}' = \frac{1}{\mathrm{Ro}} \begin{bmatrix} 0 & 1 & -\sqrt{\mathrm{Bu}}ik_x \\ -1 & 0 & -\sqrt{\mathrm{Bu}}ik_y \\ -\sqrt{\mathrm{Bu}}ik_x & -\sqrt{\mathrm{Bu}}ik_y & 0 \end{bmatrix} \begin{bmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{\eta}/\sqrt{\mathrm{Bu}} \end{bmatrix}.$$

Slow Modes

Slow mode:

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{bmatrix} \propto \begin{bmatrix} -ik_y \\ ik_x \\ 1 \end{bmatrix}$$

- note that $u = -\eta_y$ and $v = \eta_x$
- we will call this geostrophic balance

Fast Modes

Fast mode:

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{bmatrix} \propto \begin{bmatrix} -ik_y - \lambda \mathrm{Ro} ik_x \\ ik_x - \lambda \mathrm{Ro} ik_y \\ -k^2 \mathrm{Bu} \end{bmatrix} e^{i\lambda t}$$

- where here

$$\lambda = \pm \frac{1}{\mathrm{Ro}} \sqrt{\mathrm{Bu}(k_x^2 + k_y^2) + 1}.$$

- divergent phenomena: sound waves
- for $\mathrm{Ro} \ll 1$, this limits simulation timesteps...

Geostrophic Balance

- Away from the Equator: Ro $\ll 1$
- To leading order we have **geostrophic balance**

$$v = \eta_x \quad \text{and} \quad u = -\eta_y.$$

- Flow is perpendicular to the pressure gradient!
- Flow is divergence-free so no sound waves
- In dimensional coordinates

$$v = \frac{g}{f}\eta_x \quad \text{and} \quad u = -\frac{g}{f}\eta_y.$$

Shallow Water for Ro $\ll$ 1

Rossby number expansion:

$$\vec{u} = \vec{u}_0 + \vec{u}_1 \text{Ro} + \mathcal{O}(\text{Ro}^2)$$
$$\eta = \eta_0 + \eta_1 \text{Ro} + \mathcal{O}(\text{Ro}^2)$$

Geostrophic decomposition:

$$\vec{u}_i = \vec{u}_{i,g} + \vec{u}_{i,ag}, \quad \text{where} \quad \vec{u}_{i,g} \times \hat{z} = \nabla \eta_i$$

The Quasi-geostrophic Model

Then to order 0 in Ro: $\vec{u}_{0,ag} = \vec{0}$ and

$$(\vec{u}_{0,g})_t + \vec{u}_{0,g} \cdot \nabla \vec{u}_{0,g} = \vec{u}_1 \times \hat{z} - \nabla \eta_1.$$

$$(\eta_{0,g})_t + \mathrm{Bu} \nabla \cdot \vec{u}_1 = 0.$$

This simplifies to a closed equation

$$\left(1 - \frac{\Delta^{-1}}{\mathrm{Bu}}\right)(\vec{u}_{0,g})_t + \vec{u}_{0,g} \cdot \nabla \vec{u}_{0,g} = -\nabla p_{0,g},$$

where $p_{0,g}$ is the **geostrophic pressure**

$$-\Delta p_{0,g} = \nabla \cdot (\vec{u}_{0,g} \cdot \nabla \vec{u}_{0,g}).$$

Stream Function Formulation

The **stream function** ψ satisfies

$$\vec{u}_{0,g} = \widehat{z} \times \nabla\psi, \quad \text{and} \quad \nabla \times \vec{u}_{0,g} = -\Delta\psi.$$

Taking the curl of the QG-model and simplifying:

$$q_t + J(\psi, q) = 0, \quad \text{where} \quad J(\psi, q) = \psi_x q_y - \psi_y q_x,$$

$$q = \Delta\psi - \frac{1}{\text{Bu}}\psi \quad \textbf{potential vorticity}$$

Outline

A Simple Example

Consider the **KdV equation**:

$$u_t + u_{xxx} = 6uu_x.$$

Pseudo-spectral technique:

- do all products in physical space
- do all spatial derivatives in spectral space

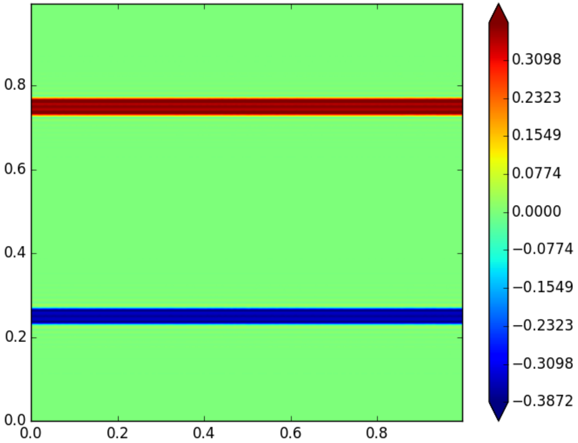
$$u_t = \mathcal{F}^{-1}(ik^3\mathcal{F}(u)) + 6u\mathcal{F}^{-1}(ik\mathcal{F}(u)).$$

- Important: aliasing errors

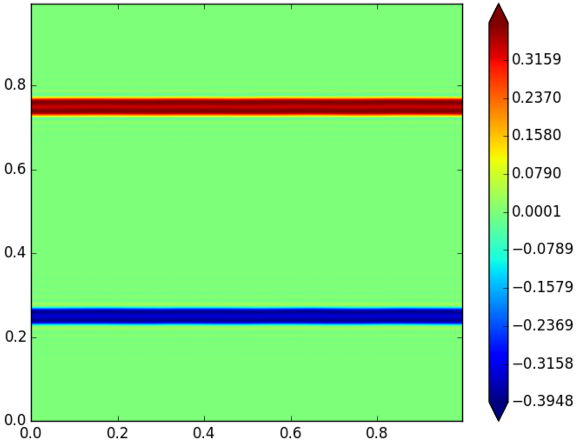
The Code

- source code written in C
- capable of solving multiple models
  - 2 and 3-dim DNS
  - Boussinesq
  - shallow water and qg shallow water
- capable of periodic and rigid lid boundary
- uses P3DFFT for 2 or 3-dim fast Fourier transforms
  - allows pencil domain decomposition
  - scales well with thousands of processors
- diagnostic calculation, file I/O, parallelization with OpenMPI

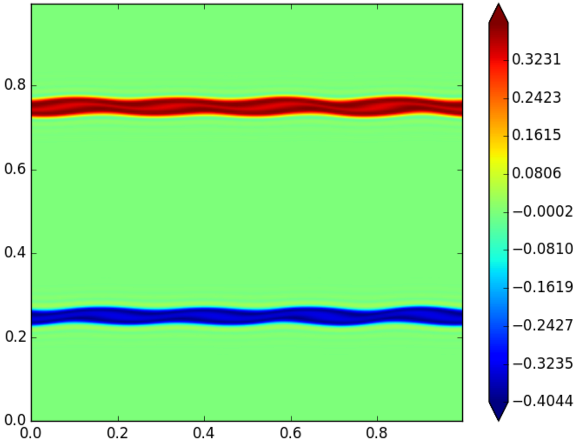
Initial PV


0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
0.3098
0.2323
0.1549
0.0774
0.0000
−0.0774
−0.1549
−0.2323
−0.3098
−0.3872

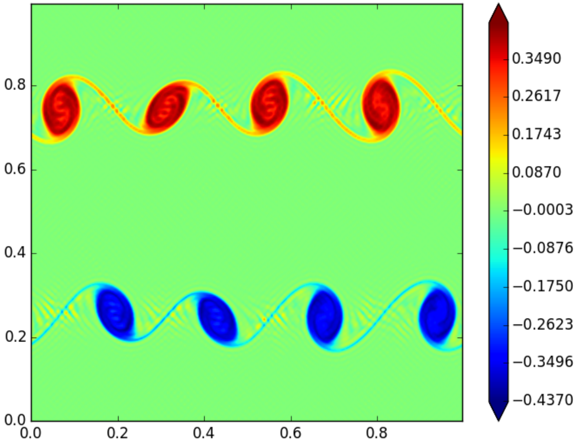
PV at $t = 50$


0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
0.3159
0.2370
0.1580
0.0790
0.0001
−0.0789
−0.1579
−0.2369
−0.3158
−0.3948

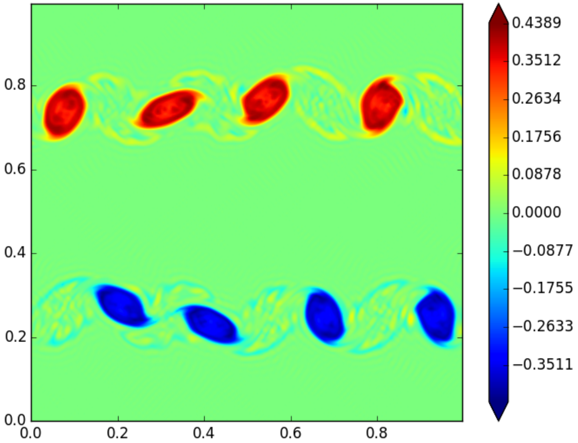
PV at $t = 100$


0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
0.3231
0.2423
0.1615
0.0806
−0.0002
−0.0810
−0.1619
−0.2427
−0.3235
−0.4044

PV at $t = 150$


0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
0.3490
0.2617
0.1743
0.0870
−0.0003
−0.0876
−0.1750
−0.2623
−0.3496
−0.4370

PV at $t = 200$


0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
0.4389
0.3512
0.2634
0.1756
0.0878
0.0000
−0.0877
−0.1755
−0.2633
−0.3511

The Evolutionary Picture

- Presence of **linear instabilities** causes growth of certain wave modes
- In above, instabilities caused by interactions between interfacial waves
- The fastest growing mode will be dominant
- Nonlinear advection term causes these to rotate/interact
- Eventually transitions to turbulent regime (dipole pv)

Questions

- Energy flows from small scales to large scales (inverse scattering)
- Can we predict the number of vortices?
- How does the number of vortices vary with Ro, Fr, Bu?
- How does the picture change in the full shallow water equations?

Outline

Linearized QG

Consider a perturbation ψ^p from a background profile ψ^b:

$$q_t^b + q_t^p + J(\psi^b, q^b) + J(\psi^b, q^p) + J(\psi^p, q^b) + J(\psi^p, q^p) = 0,$$

Dropping nonlinear perturbation terms:

Linearized QG PV Equation

$$q_t^p + J(\psi^b, q^p) + J(\psi^p, q^b) = 0$$

Rayleigh Equation

Base state:

$$\psi^b = -\int u^b(y)dy$$

Linearized QG PV:

$$q_t^p + (q^b)'\psi_x^p - (\psi^b)'q_x^p = 0.$$

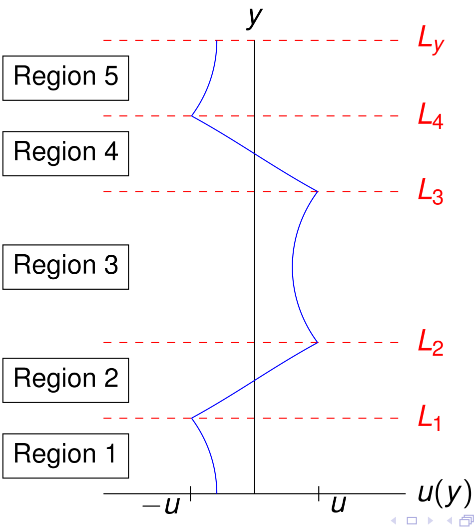
Substitute perturbed solution:

$$\psi^p(x,y,t) = e^{ik(x-ct)}f(y)$$

Rayleigh Equation:

$$f''(y) - \left(k^2 + \frac{u^b(y)'' - c/\mathrm{Bu}}{u^b(y) - c}\right)f(y) = 0.$$

Background Profile



Background Profile

$$
\begin{aligned}
u^b(y) &= -b\cosh(y/\sqrt{\text{Bu}}), && 0 \le y < L_1 \\
u^b(y) &= m\sinh\left(\frac{y - L_y/4}{\sqrt{\text{Bu}}}\right), && L_1 \le y < L_2 \\
u^b(y) &= b\cosh\left(\frac{y - L_y/2}{\sqrt{\text{Bu}}}\right)), && L_2 \le y < L_3 \\
u^b(y) &= -m\sinh\left(\frac{y - 3L_y/4}{\sqrt{\text{Bu}}}\right), && L_3 \le y < L_4 \\
u^b(y) &= -b\cosh\left(\frac{y - L_y}{\sqrt{\text{Bu}}}\right), && L_4 \le y < L_y \\
m &= \frac{u}{\sinh(d/2\sqrt{\text{Bu}})} \\
b &= \frac{u}{\cosh(L_1/\sqrt{\text{Bu}})}
\end{aligned}
$$

Analytic solution with Rayleigh

$$f_i(y) = A_i e^{\tilde{k}y} + B_i e^{-\tilde{k}y},\ i = 1, \ldots, 5.$$

for $\tilde{k} = \sqrt{k^2 + \frac{1}{\mathrm{Bu}}}$. **Jump conditions:**

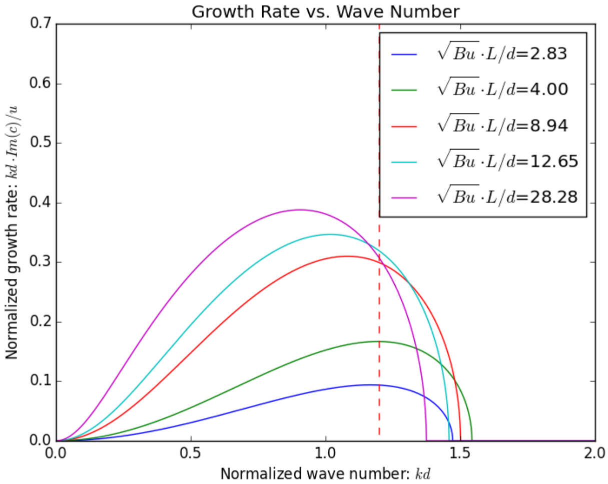
$$W(f(y), u^b(y) - c),\ \text{ and } \frac{f(y)}{u^b(y) - c} \text{ are continuous at interfaces.}$$

Result in a system of linear equations for A_i, B_i.
Approx. Scattering Relation:

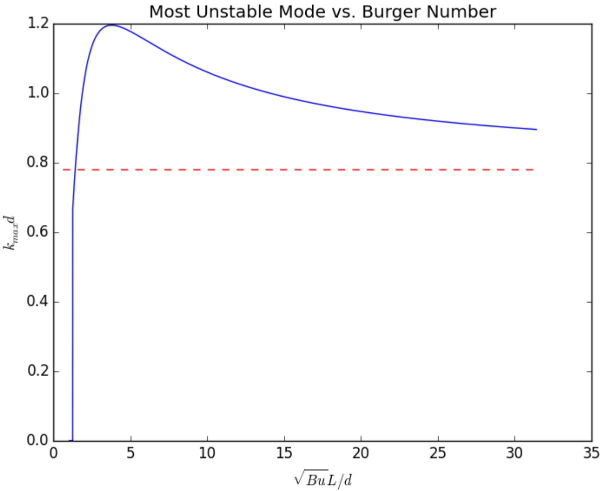
$$\frac{c^2 k^2 d^2}{u^2} = \left[kd - z\frac{kd}{\sqrt{(kd)^2 + d^2/\mathrm{Bu}}}\right]^2 - e^{-2\tilde{k}d}\frac{(kd)^2}{(kd)^2 + d^2/\mathrm{Bu}} z^2$$

for $z = (d/\sqrt{\mathrm{Bu}})(1 + \coth(d/2\sqrt{\mathrm{Bu}}))/2$.

Instability Growth Rate



Most Unstable Wave Number



Modes are eventually stable

Theorem (Casper)

Let y_i be an inflection point of $u^b(y)$ and set $u_s = u^b(y_i)$. Let k_s^2 be the maximum eigenvalue of the discrete spectrum of

$$D(y, \partial_y) = \partial_y^2 - \frac{1}{Bu} + K(y), \quad K(y) = -\frac{(u^b)''(y) - u^b(y)/Bu}{u^b(y) - u_s}.$$

with $K(y) \geq 0$ throughout the field. Then k is stable for $k \geq k_s$.

Outline

Growth Rates

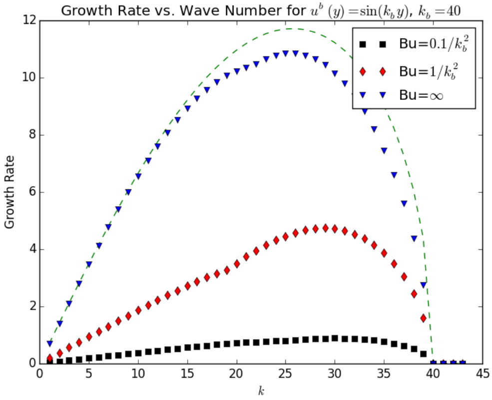


Figure: Growth rate vs. wave number for $u^b(y) = \sin(40y)$

Growth Rates

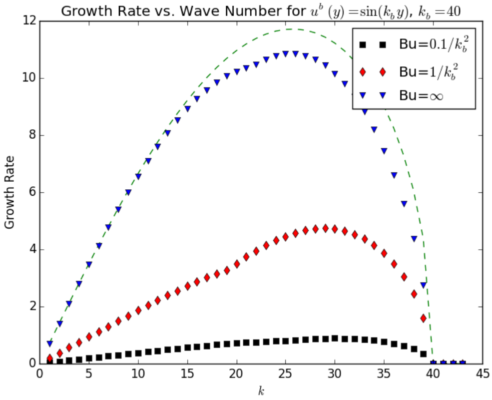


Figure: Must be stable for $k \geq 40$ and move toward stability as $k \to 40$

Growth Rates

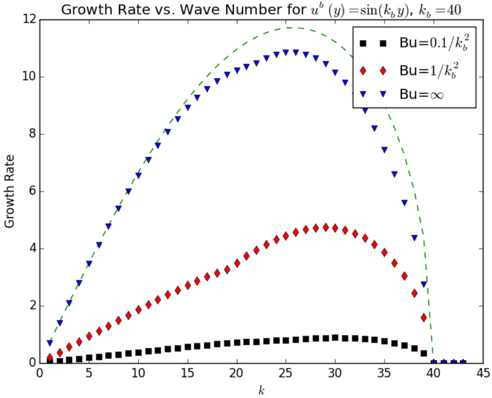


Figure: The most unstable wave number increases with decreasing Bu. The dashed green line is an analytic approximation for Bu $= \infty$.

Outline

Linearization of Shallow Water

Background profile:

$$\eta^b = \eta^b(y), \quad u^b = u^b(y) = -\eta^b(y)', \quad v^b = 0.$$

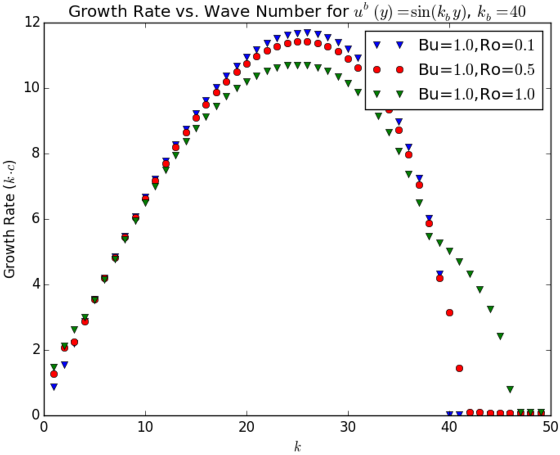
Separable solution:

$$\eta^p = e^{ik(x-ct)}\widetilde{\eta}(y), \quad u^p = e^{ik(x-ct)}\widetilde{u}(y), \quad v^p = e^{ik(x-ct)}i\widetilde{v}(y)$$

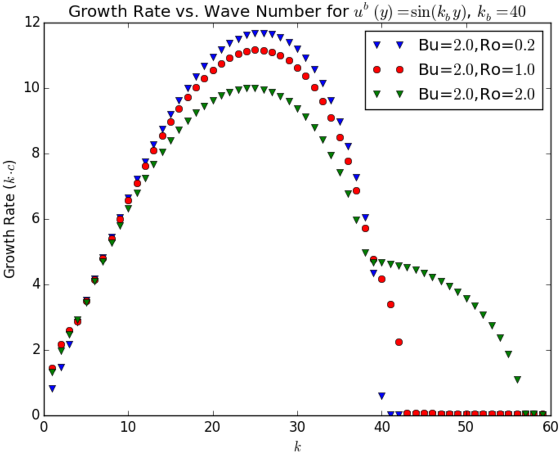
Linearized equation:

$$c\begin{bmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{\eta}/\sqrt{\mathrm{Bu}} \end{bmatrix} = \frac{1}{\mathrm{Ro}}\begin{bmatrix} u^b\mathrm{Ro} & -1/k + \frac{(u^b)'}{k}Ro & \sqrt{\mathrm{Bu}} \\ -1/k & u^b\mathrm{Ro} & -\frac{\sqrt{\mathrm{Bu}}}{k}\partial \\ \sqrt{\mathrm{Bu}} + \frac{\mathrm{Ro}}{\sqrt{\mathrm{Bu}}}\eta^b & \frac{\sqrt{\mathrm{Bu}}}{k}\partial + \frac{\mathrm{Ro}}{k\sqrt{\mathrm{Bu}}}(\eta^b\partial + (\eta^b)') & u^b\mathrm{Ro} \end{bmatrix}\begin{bmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{\eta}/\sqrt{\mathrm{Bu}} \end{bmatrix}$$

Growth Rates



Growth Rates



Ageostrophic Instabilities

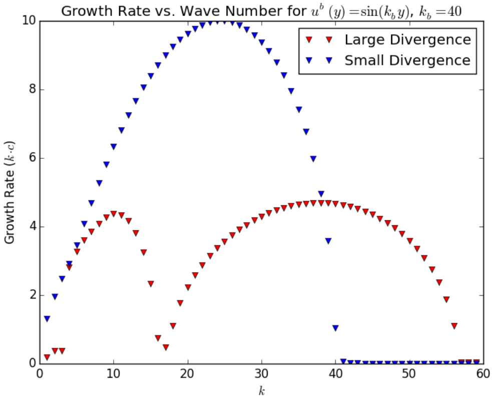


Figure: The spectrum is broken up by high/low divergence.

Outline

Orthogonal Polynomials on $\mathbb{R}$

$\mu(x)$ a measure on $\mathbb{R}$ with finite moments.

Definition

The (normalized) **sequence of orthogonal polynomials** (SOP) $p_0(x), p_1(x), \ldots$ for μ satisfies

- $\int p_m(x)p_n(x)d\mu(x) = 0$ for $m \neq n$
- $\deg(p_n(x)) = n$ for all n
- $\int p_n(x)^2 d\mu(x) = 1$

Jacobi Matrices

Theorem

If $p_0(x), p_1(x), \ldots$ is a SOP for μ, then

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x).$$

$$\begin{bmatrix} b_0 & a_0 & 0 & 0 & \ldots \\ a_0 & b_1 & a_1 & 0 & \ldots \\ 0 & a_1 & b_2 & a_2 & \ldots \\ 0 & 0 & a_2 & b_2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{bmatrix} = x \begin{bmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{bmatrix}.$$

Outline

Rayleigh's Equation

For $u^b(y) = \cos(y)$, Rayleigh's equation

$$f''(y) - \left(k^2 + \frac{-\cos(y) - c/\text{Bu}}{\cos(y) - c}\right) f(y) = 0.$$

In spectral space:

$$\frac{1}{2}\frac{(\ell+1)^2 + k^2 - 1}{\ell^2 + k^2 + 1/\text{Bu}}\hat{f}(\ell+1) + \frac{1}{2}\frac{(\ell-1)^2 + k^2 - 1}{\ell^2 + k^2 + 1/\text{Bu}}\hat{f}(\ell-1) = c\hat{f}(\ell),$$

Rayleigh's Equation

Setting

$$\hat{q}(\ell) = \frac{1}{\sqrt{2}} \left(\frac{\ell^2 + k^2 - 1}{\ell^2 + k^2 + 1/\mathrm{Bu}} \right)^{1/2}$$

$$\hat{g}(\ell) = (\ell^2 + k^2 - 1)^{1/2} (\ell^2 + k^2 + 1/\mathrm{Bu})^{1/2} \hat{f}(\ell)$$

We have

$$c\hat{g}(\ell) = \hat{q}(\ell)\hat{q}(\ell+1)\hat{g}(\ell+1) + \hat{q}(\ell)\hat{q}(\ell-1)\hat{g}(\ell-1).$$

For convenience, let

$$z_j = \widehat{q}(j)\widehat{q}(j+1)$$

Eigenvectors of bi-infinite matrix

$$B = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & 0 & z_1 & 0 & 0 & 0 & \dots \\ \dots & z_1 & 0 & z_0 & 0 & 0 & \dots \\ \dots & 0 & z_0 & 0 & z_0 & 0 & \dots \\ \dots & 0 & 0 & z_0 & 0 & z_1 & \dots \\ \dots & 0 & 0 & 0 & z_1 & 0 & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Eigenvectors of Jacobi matrix

$$\widetilde{B} = \begin{bmatrix} 0 & 2z_0 & 0 & 0 & \dots \\ z_0 & 0 & z_1 & 0 & \dots \\ 0 & z_1 & 0 & z_2 & \dots \\ 0 & 0 & z_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Eigenvectors of Jacobi matrix

$$\widetilde{B}^2 = \begin{bmatrix} 2z_0^2 & 0 & 2z_0z_1 & 0 & \dots \\ 0 & 2z_0^2 + z_1^2 & 0 & z_1z_2 & \dots \\ z_0z_1 & 0 & z_2^2 + z_2^2 & 0 & \dots \\ 0 & z_1z_2 & 0 & z_2^2 + z_3^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Eigenvectors of Jacobi matrix

$$A = \begin{bmatrix} 2z_0^2 + z_1^2 & z_1 z_2 & 0 & 0 & \dots \\ z_1 z_2 & z_2^2 + z_3^2 & z_3 z_4 & 0 & \dots \\ 0 & z_3 z_4 & z_4^2 + z_5^2 & z_5 z_6 & \dots \\ 0 & 0 & z_5 z_6 & z_6^2 + z_7^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Outline

Spectrum of A

Theorem (Casper)

The sequence of orthogonal polynomials $p_n(x)$ associated to A satisfies:

- *each $p_n(x)$ has exactly one negative root r_{n1}*
- *$-r_{n1}$ is monotone increasing*
- *$-r_{n1}$ converges to the unique negative eigenvalue of A*
- *the limit is the growth rate*

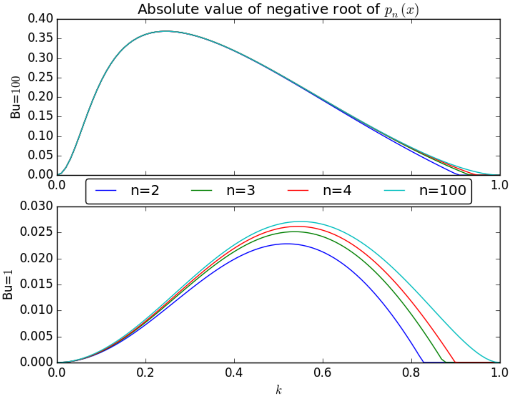


Figure: $|r_{n1}|$ vs k for various values of n at Burger number 1 and 100. Convergence is slower for larger Burger.

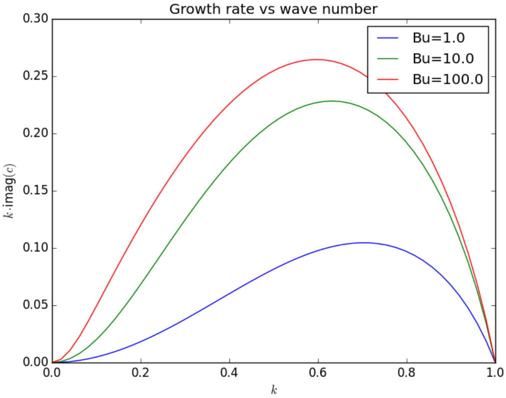


Figure: The growth rate vs. wave number for various values of the Burger number. Predicts increasing max unstable wave number with decreasing Burger.

Thank You!

- Casper, W. R. *A Connection Between Orthogonal Polynomials and Shear Instabilities in the Quasi-geostrophic Shallow Water Equations*. ArXiv 1701.07048
- Nadiga, B. T. *Nonlinear Evolution of a Baroclinic Wave and Imbalanced Dissipation* J. Fluid Mech. (2014), vol 756, pp. 965-1006
- Drazin, P. G. and Reid, W. H. *Hydrodynamic Stability* New York: Cambridge University Press. ISBN 978-0521227988.