Representation Theory and the Matrix Bochner Problem

AMS-CMS Joint Meeing

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Outline

Basic definitions

Definition

A **weight matrix** $W(x)$ **supported on** (a, b) is an $N \times N$ matrix-valued function on $\mathbb{R}$ which:

- is smooth, positive-definite, and Hermitian on (a, b)
- is zero on $\mathbb{R} \backslash (a, b)$
- has finite moments $\int_a^b |x|^n W(x) dx < \infty$

Example:

$$W(x) = \begin{pmatrix} 1 + |a|^2 x^2 & ax \\ \bar{a}x & 1 \end{pmatrix} e^{-x^2}$$

Basic definitions

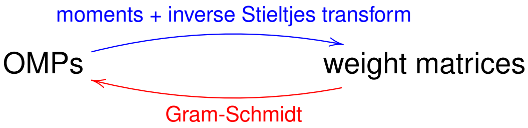
Definition

A sequence of **orthogonal matrix polynomials** (OMPs) on $\mathbb{R}$ is a sequence of $N \times N$ matrix-valued polynomials $P_0(x), P_1(x), \ldots$ with

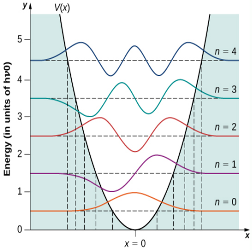
- $P_n(x)$ degree n with nonsingular leading coeff. for all n
- $\int P_m(x)W(x)P_n(x)^* dx = 0$ for $m \neq n$

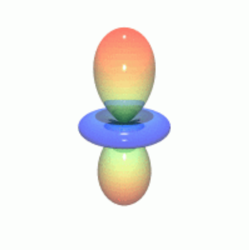
where $W(x)$ is an $N \times N$ weight matrix on $\mathbb{R}$.

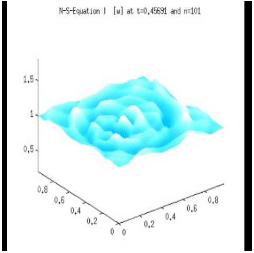
Correspondence:

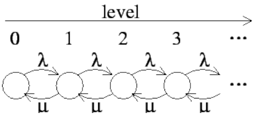


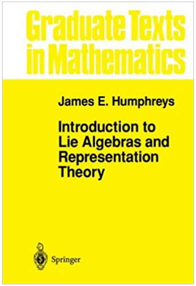
Applications

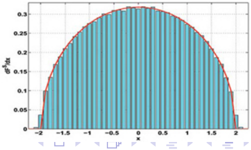












Matrix Differential Operators

- We will consider matrix-valued differential operators

Example:

$$\mathfrak{D} = \partial_x^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \partial_x \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} \partial_x x + 1 & \partial_x^2 \\ 2 & 0 \end{pmatrix}.$$

Right action:

$$P(x) \cdot \mathfrak{D} = P''(x) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + P'(x) \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + P(x) \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}.$$

Matrix Bochner Problem

Problem

Classify all orthogonal matrix polynomials which are eigenfunctions of a second-order matrix differential operator.

- Equiv. classify all weights $W(x)$
- Long-standing (Bochner 1929, Kreĭn 1949, Durán 1997)
- More generally, calculate

$$\mathcal{D}(W) = \{\mathfrak{D} : \forall n \exists \Lambda(n) \in M_N(\mathbb{C}) \text{ s.t. } P_n(x) \cdot \mathfrak{D} = \Lambda(n) P_n(x)\}.$$

Classical (scalar) examples

- **Hermite polynomials** corresp. to weight $W(x) = e^{-x^2}$ are eigenfunctions of $\partial_x^2 - 2x$
- **Laguerre polynomials** corresp. to weight $W(x) = x^b e^{-x} 1_{(0,\infty)}(x)$ are eigenfunctions of $\partial_x^2 x + \partial_x(b + 1 - x)$
- **Jacobi polynomials** corresp. to weight $W(x) = (1 - x)^a (1 + x)^b 1_{(-1,1)}(x)$ are eigenfunctions of $\partial_x^2 (1 - x)^2 + \partial_x(a - b - (a + b + 2)x)$

Theorem (Bochner)

Up to affine trans. these are all possible cases which are 1×1

Differential operators and algebras

Idea

The value of a differential operator $\mathfrak{D}$ is controlled by the algebras they are contained in.

Example ($\mathfrak{d} = \partial_x^2 + u(x)$):

Centralizer $\mathcal{C}(\mathfrak{d})$ can determine $u(x)$.

- If $\mathcal{C}(\mathfrak{d})$ contains an operator of order 3, then $u(x)$ gives a soliton solution of KdV
- More generally, if $X = \text{Spec}(\mathcal{C}(\mathfrak{d})) \not\cong \mathbb{A}^1$ then $u(x)$ is determined by a special function related to X

The algebra $\mathcal{D}(W)$

Big idea: determine $\mathfrak{D}$ from the algebraic structure of $\mathcal{D}(W)$

- $\mathcal{D}(W) \subseteq M_N(\mathbb{C}[x, \partial_x])^{op}$ noncommutative
- closed under adjoint operation

$$\mathfrak{D} \mapsto \mathfrak{D}^\dagger = W(x)\mathfrak{D}^* W(x)^{-1}.$$

- center $\mathcal{Z}(W)$ is reduced, affine over $\mathbb{C}$, Krull dim 1
- $\mathcal{D}(W)$ is also affine algebra over $\mathbb{C}$ (nontrivial!)
- $\mathcal{D}(W)$ is a finite $\mathcal{Z}(W)$-module
- noncommutative, semiprime PI algebra
- bispectral algebra

Local structure of $\mathcal{D}(W)$

- Ring of fractions

$$\mathcal{F}(W) = \{B^{-1}A : A, B \in \mathcal{Z}(W),\ B \text{ not a zero divisor}\}.$$

- $\mathcal{F}_i(W)$, $i = 1, \dots, r$ fraction field of i'th irred. component of Spec($\mathcal{Z}(W)$)

Theorem (-,Yakimov 2018)

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i(W)).$$

Classification

- $n_1 + \cdots + n_r$ is the **rank** of $\mathcal{D}(W)$ (bounded by ℓ)

Theorem (-,Yakimov 2018)

Let $W(x)$ be an $\ell \times \ell$ weight matrix solving Bochner, with $\mathcal{D}(W)$ having rank ℓ. Then $W(x)$ is a noncommutative bispectral Darboux transformation of a direct sum of classical weights. In particular

$$W(x) = T(x)diag(r_1(x), \dots, r_\ell(x))T(x)^*$$

$$P_n(x) = diag(p_{1n}(x), \dots, p_{\ell,n}(x)) \cdot \mathfrak{T}$$

for some rational matrix $T(x)$, classical weights $r_1(x), \dots, r_\ell(x)$, and differential operator $\mathfrak{T}$.

Thanks for listening!

- New paper: https://arxiv.org/abs/1803.04405

- Bochner, Salomon. Über Sturm-Liouvillesche Polynomsysteme, Mathematische Zeitschrift 1929.
- Kreĭn, M. Infinite *J*-matrices and the matrix-moment problem, DokladyAkad. Nauk SSSR 1949
- Geiger, Joel and Horozov, Emil and Yakimov, Milen. Noncommutative bispectral Darboux transformations, Transactions AMS 2017
- Koelink, Erik and van Pruijssen, Maarten and Román, Pablo. Matrix-valued orthogonal polynomials related to $(SU(2) \times SU(2), \text{diag})$, IMRN 2012