# The Geometry of Commuting Integral and Differential Operators

California State University Fullerton, January 2020

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# Outline

1. Commuting Integral and Differential Operators
   - Time and Band-Limiting
   - Bispectrality
2. Proving the Conjecture
   - Geometry of Differential Operators
   - Adjoints of Differential Operators
   - Sketch of Proof
3. Future Directions
   - Discrete Time and Band Limiting
   - Other Future Directions

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# A Problem from Mathematical Communication Theory

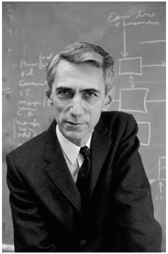

Figure: Claude Shannon

**Problem (Shannon 1948)**

What is the best quality of information that can be conveyed over a phone line?

## Our Mathematical Model

1. information: function of time $f(t)$
2. convert to amplitudes and frequencies to send over the line
3. convert back at the other end

Noise limits the frequencies $[-\kappa, \kappa]$ we can communicate, and the call duration $\tau$ limits the time.

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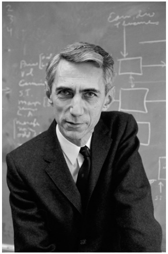

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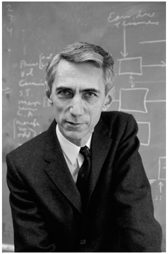

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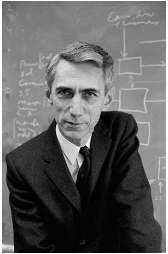

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# Integral Operators

Our model is expressed mathematically as an integral operator.

**Definition**

A **integral operator** is a transformation $T$ taking a function $f(x)$ to a new function

$$T(f)(x) = \int_a^b K(x,y)f(y)dy,$$

where here $K(x,y)$ is a function of two variables called the **kernel** of $T$.

Important examples: Fourier transform and Laplace transform

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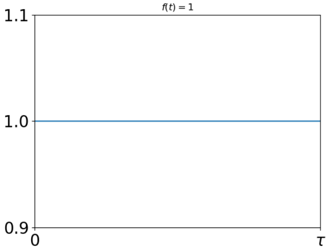
# Time and Band-Limiting Operator

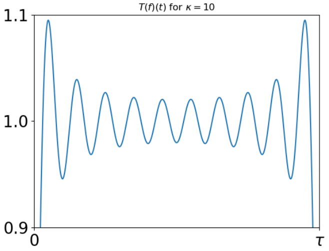
Communicating $f(t)$ over the phone line is the same as:

$$T(f)(t) = \int_0^\tau \frac{\sin(2\pi\kappa(t-s))}{\pi(t-s)} f(s)ds, \quad 0 \leq t \leq \tau.$$

This is called a **time and band-limiting operator.**




(a) $f(t) = 1$




(b) $T(f)(t)$, when $\kappa = 10/\tau$

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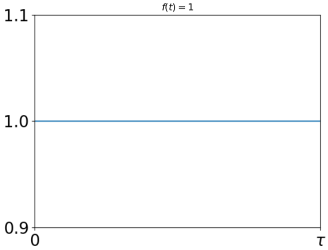


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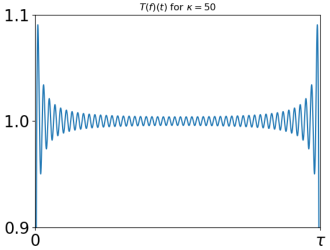


(b) $T(f)(t)$, when $\kappa = 50/\tau$

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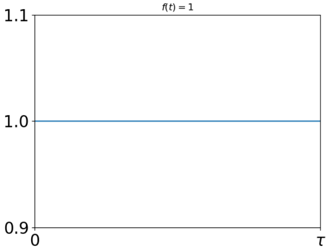


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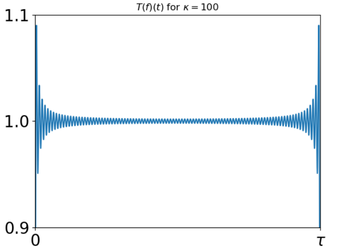


(b) $T(f)(t)$, when $\kappa = 100/\tau$

# Shannon's Problem

**Problem (Shannon)**

Can we come up with a way to recover $f$ back from $T(f)$ for some functions?

- Yes, for eigenfunctions of $T$!

**Definition**

An **eigenfunction** of $T$ is a function $f(t)$ satisfying

$$T(f)(t) = \lambda f(t), \quad \text{for some} \quad \lambda \in \mathbb{C}.$$

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# Fast-forward 20 years

- the problem is solved by Landau, Pollak, and Slepian!

**Definition**

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is a transformation taking a function $f(x)$ to a new function

$$D(f)(x) = a_0(x)f(x) + a_1(x)f'(x) + a_2(x)f''(x) + \cdots + a_n(x)f^{(n)}(x).$$

**Example**

$D(x, \partial_x) = x\partial_x^2 + 2x$ means $D(f)(x) = xf''(x) + 2xf(x)$.

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# Commuting Operators

**Theorem (Landau-Pollak and Pollak-Slepian)**

*The differential operator $D(x, \partial_x) = (\kappa^2 - x^2)\partial_x^2 - 2x\partial_x + \tau^2 x^2$ **commutes** with Shannon's time and band-limiting operator $T$:*

$$T(D(f))(x) = D(T(f))(x).$$

- Consequence: $T$ and $D$ will have to share eigenfunctions.

$$T(f)(x) = \lambda f(x) \text{ if and only if } D(f)(x) = \mu f(x)$$

- Just solve the **differential equation**

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# Outline

# More Examples are Found

**Question**

More commuting integral and differential operators?

**Slepian 1960's**

1. examples from spherical harmonics
2. extensions to $n$ dimensions

**Tracy and Widom 1990's**

1. examples from random matrix theory
2. defined by Airy and Bessel functions
3. 2020 Steele prize

Figure: David Slepian

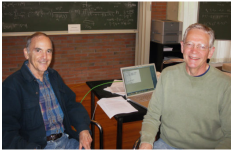

Figure: Craig Tracy and Harold Widom

# Unifying Theory

- Examples arise naturally in diverse areas!

**Question**

Can we find a unifying theory or construction?

Observation by Duistermaat and Grünbaum (1986):

- the integral operators found all have very special kernels

$$K(x,y) = \int_{-r}^{r} \psi(x,z)\psi^*(y,z)dz$$

where $\psi(x,z)$ is a special kind of function called a **bispectral function.**

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# Bispectral functions

## Definition

A function $\psi(x, z)$ is **bispectral** if it simultaneously satisfies two differential equations

$$a_0(x)\psi + a_1(x)\frac{\partial\psi}{\partial x} + \cdots + a_m(x)\frac{\partial^m\psi}{\partial x^m} = g(z)\psi.$$

$$b_0(z)\psi + b_1(z)\frac{\partial\psi}{\partial z} + \cdots + b_n(z)\frac{\partial^n\psi}{\partial z^n} = f(x)\psi.$$

In terms of differential operators:

- there are operators $D(x, \partial_x)$ and $B(z, \partial_z)$ with

$$D(x, \partial_x) \cdot \psi(x, z) = g(z)\psi(x, z)$$

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# Bispectral Examples

## Example

The function $\psi(x,z) = e^{xz}$ is bispectral since

$$\frac{\partial\psi}{\partial x} = z\psi(x,z) \text{ and } \frac{\partial\psi}{\partial z} = x\psi(x,z).$$

## Example

The function $\psi(x,z) = e^{xz}(1 - x^{-1}z^{-1})$ is bispectral since

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# Bispectral Examples

The Airy funtion $\mathrm{Ai}(x)$ satisfies the Airy differential equation

$$\mathrm{Ai}''(x) = x\mathrm{Ai}(x).$$

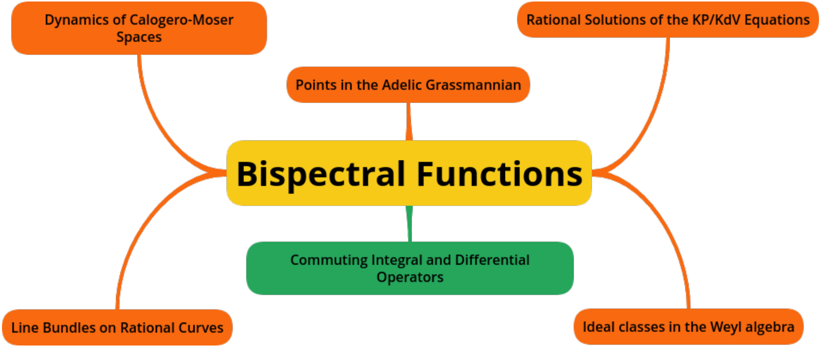
## Example

The function $\psi(x,z) = \mathrm{Ai}(x+z)$ is bispectral since

$$\frac{\partial^2\psi}{\partial x^2} - x\psi = z\psi \quad \text{and} \quad \frac{\partial^2\psi}{\partial z^2} - z\psi = x\psi$$

# Interpretations of Bispectrality

Yuri Berest, Igor Krichever, George Wilson, ...



# Unifying Theory

## Conjecture (Duistermaat-Grünbaum 1986)

*For any sufficiently nice bispectral function* $\psi(x, z)$ *the integral operator*

$$T(f)(x) = \int_{-s}^{s} K(x, y) f(y) dy$$

*with kernel*

$$K(x, y) = \int_{-r}^{r} \psi(x, z) \psi^*(y, z) dz$$

*commutes with a nonconstant differential operator.*

# Outline

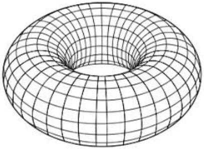
# The Spectral Curve

## Definition

The **eigenvalues** of a differential operator $D(x, \partial_x)$ are the complex numbers $\lambda$ with

$$D(x, \partial_x) \cdot f(x) = \lambda f(x), \quad \text{for some nonzero } f(x).$$

$$D(x, \partial_x)$$
$$\downarrow$$
$$\{\text{eigenvalues of } D(x, \partial_x)\} \rightarrow$$



**Spectral Curve**
(compact, complex surface)

# The Spectral Curve

**Definition**

$D(x, \partial_x)$ is **bispectral** if for some bispectral $\psi(x, z)$

$$D(x, \partial_x) \cdot \psi(x, z) = g(z)\psi(x, z).$$

$D(x, \partial_x)$ bispectral

$\downarrow$

$\{\text{eigenvalues of } D(x, \partial_x)\} \rightarrow$

Balloon-like surface
(sphere with pinched point(s))

# Explicit Construction

Take $D(x, \partial_x)$ a differential operator.

- Schur 1905: the centralizer $Z(D)$ of $D$ is commutative
- Burchnall-Chaundy 1929: commuting differential operators are algebraically dependent
- Using filtration by order to define

$$C = \text{Proj}(\text{Rees}(Z(D))) = \text{Spec}(Z(D)) \cup \{\infty\},$$

$$\text{Rees}(Z(D)) = \bigoplus_{n \geq 0} \{L \in Z(D) : \text{order}(L) \leq n\} t^n.$$

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# Differential Operators on Spectral Curves

Big idea: consider differential operators on the spectral curve!

**Definition**

Let $C$ be a spectral curve and let

$$\mathcal{A} = \{\text{holomorphic functions } f : C\backslash\{\infty\} \to \mathbb{C}\}.$$

A **differential operator** on $C$ is a transformation

$$R : \mathcal{A} \to \mathcal{A},\ \ f(z) \mapsto R(f)(z)$$

which satisfies the Ad-condition.

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# Ad-condition

Each $a(z) \in \mathcal{A}$ defines a differential operator

$$M_a : f(z) \mapsto a(z)f(z).$$

Observation: if $R = R(z, \partial_z)$ is a differential operator

$$\text{order}(\text{Ad}_{M_a}^k(R)) \leq \text{order}(R) - k.$$

**Definition**

A linear transformation $R : \mathcal{A} \to \mathcal{A}$ satisfies the Ad-**condition** if there exists $k > 0$ with $\text{Ad}_{M_a}^{k+1}(R) = 0$ for all $a \in \mathcal{A}$.

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$\psi(x, z)$ bispectral with operator $D(x, \partial_x)$ and spectral curve $C$

**Theorem (Casper et al.)**

*Let $R = R(z, \partial_z)$ be a differential operator on $C$. Then there exists $L(x, \partial_x)$ with*

$$L(x, \partial_x) \cdot \psi(x, z) = R(z, \partial_z) \cdot \psi(x, z).$$

Define the **left and right Fourier algebras**:

$$\mathcal{F}_x(\psi) = \{D(x, \partial_x) : \text{ there exists } B(z, \partial_z) \text{ with } B \cdot \psi = D \cdot \psi\}.$$

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Define the **left and right Fourier algebras**:

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# Differential Operators on Spectral Curves

$\psi(x, z)$ bispectral with operator $D(x, \partial_x)$ and spectral curve $C$

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# Fourier algebra example

Consider the bispectral function $\psi(x,z) = e^{xz}$.

- $x\partial_x \in \mathcal{F}_x(\psi)$ because

$$x\partial_x \cdot \psi(x,z) = xze^{xz} = z\partial_z \cdot \psi(x,z).$$

- in fact $x^m\partial_x^n \in \mathcal{F}_x(\psi)$ for all $m, n > 0$ because

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# $\mathcal{F}_x(\psi)$ is Really Big

**Theorem (Casper et al.)**

*The subspace*

$$\mathcal{F}_x^{\ell,m}(\psi) = \{L(x,\partial_x) : B\cdot\psi = D\cdot\psi,\ \ \text{order}(L) \leq \ell,\ \ \text{order}(R) \leq m\}$$

*has dimension*

$$\dim(\mathcal{F}_x^{\ell,m}(\psi)) \geq (\ell+1)(m+1) - 2g_{diff}.$$

# Outline

1. Commuting Integral and Differential Operators
   - Time and Band-Limiting
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# Integration by Parts

Remember integration by parts:

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b + \int_a^b -f'(x)g(x).$$

A more complicated example:

$$\int_a^b f(x)g''(x)dx = f(x)g'(x)|_a^b - \int_a^b f'(x)g'(x)dx$$

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# Adjoints of Differential Operators

## Definition

For any differential operator

$$D(x, \partial_x) = a_0(x) + a_1(x)\partial_x + a_2(x)\partial_x^2 \cdots + a_n(x)\partial_x^n$$

The **formal adjoint** is

$$D^*(x, \partial_x) = a_0(x) - \partial_x a_1(x) + \partial_x^2 a_2(x) + \cdots + (-1)^n \partial_x^n a_n(x).$$

For example, if $D(x, \partial_x) = x^2\partial_x$ then

$$D^*(x, \partial_x) = -\partial_x x^2 = -x^2\partial_x - 2x.$$

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# Super Integration by Parts

If $D(x, \partial_x)$ is a differential operator

$$\int_a^b f(x)D(x,\partial_x)\cdot g(x)dx = C_D(f,g;b) - C_D(f,g;a) + \int_a^b g(x)D^*(x,\partial_x)\cdot f(x)dx$$

Here $C_D(f, g; b)$ is the **bilinear concomitant**, defined by:

- $D(x, \partial_x)$
- the derivatives of $f(x)$ and $g(x)$ at the point $b$

# Outline

# Main Theorem

**Theorem (Casper-Yakimov 2019)**

*Let $\psi(x,z)$ be a self-adjoint, rank 1 bispectral function. Then the integral operator*

$$T(f)(x) = \int_{-s}^{s} K(x,y)f(y)dy,$$

*with kernel*

$$K(x,y) = \int_{-r}^{r} \psi(x,z)\psi(y,z)dz$$

*commutes with a nonconstant differential operator in $\mathcal{F}_x(\psi)$.*

# Proof Sketch

- Choose $D(x, \partial_x)$ and $B(z, \partial_z)$ with $D \cdot \psi = B \cdot \psi$ so
  - they are **self-adjoint**:

$$D(x, \partial_x) = D^*(x, \partial_x) \text{ and } B(z, \partial_z) = B^*(z, \partial_z)$$

  - the concomitants of $D$ vanish

$$C_D(f, g; \pm s) = 0 \text{ for all } f(x), g(x)$$

  - the concomitants of $B$ also vanish

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- Use Super Integration by Parts!

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D(x,\partial_x)\cdot K(x,y) &= \int_{-r}^{r} D(x,\partial_x)\cdot\psi(x,z)\psi(y,z)dz \\
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# Proof Sketch

- Use Super Integration by Parts again!

$$
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D(x,\partial_x)\cdot T(f)(x) &= \int_{-s}^{s} D(x,\partial_x)\cdot K(x,y)f(y)dy \\
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# Discrete Examples

**Idea:**

- replace $T$ with a matrix which acts like an integral operator

$$N \times N \text{ Hankel matrix } H_{ij} = h(i+j).$$

- replace $D$ with a matrix which acts like a differential operator

$$N \times N \text{ tri-diagonal } B_{ij} = 0 \text{ for } |i-j| > 1.$$

**Question**

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# Hilbert Matrix Example

The $N \times N$ **Hilbert matrix** is

$$H_{ij} = \frac{1}{i + j + \mu}, \quad 1 \leq i, j \leq N.$$

It commutes with a special tridiagonal matrix

$$B_{ij} = \begin{cases} -2(N-i)(N+i+\lambda)(i^2 + (i-1)\lambda - n), & i = j \\ i(N-i)(1+i+\lambda)(N+1+i+\lambda), & j = i+1 \\ B_{ji}, & i = j+1 \end{cases}$$

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# Application: Eigenvectors of the Hilbert Matrix

**Problem**

Find the **eigenvectors** of the $N \times N$ Hilbert matrix $H$:

$$\text{find } \vec{v} \text{ with } \; H\vec{v} = \lambda\vec{v} \; \text{ for some } \vec{v}.$$

Numerically ill-posed!

- Idea: $H$ and $B$ have the same eigenvectors
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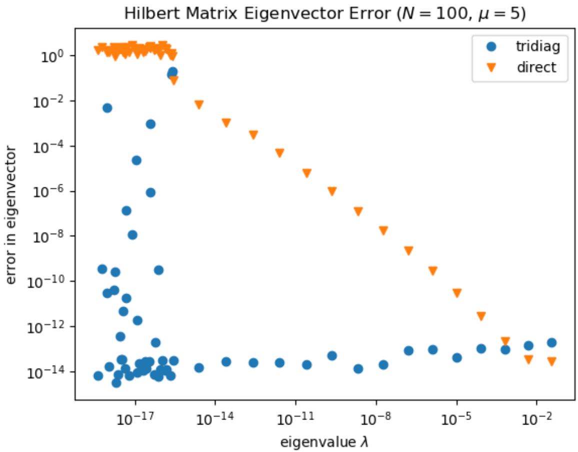

# Outline

# Future Work

1. numerical approximation of eigenfunctions for integral operators
2. dynamics of Calogero-Moser spaces
3. orthogonal polynomials
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6. derived equivalence

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# Thank You!


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