Modeling with First-Order Equations

California State University Fullerton, February 2020

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February 2, 2020

First-Order Models

Many real-world situations are **modeled** by first-order ordinary differential equations

$$\frac{dy}{dt} = f(t, y).$$

Examples:

- interest rates
- mixing fluids in a tank
- population dynamics
- falling bodies

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Red Bull Stratos Skydive

Figure: Felix Baumgartner jumps from the stratosphere

Breaking records

- Maximum Vertical Speed!

  843.6 mph or Mach 1.25

- Highest jump altitude!

  127852.4 feet or 24.21 miles

- Longest freefall distance!

  119431.1 feet or 22.619 miles

- First freefall to break the speed of sound
- Highest untethered altitude outside a vehicle

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Velocity versus time

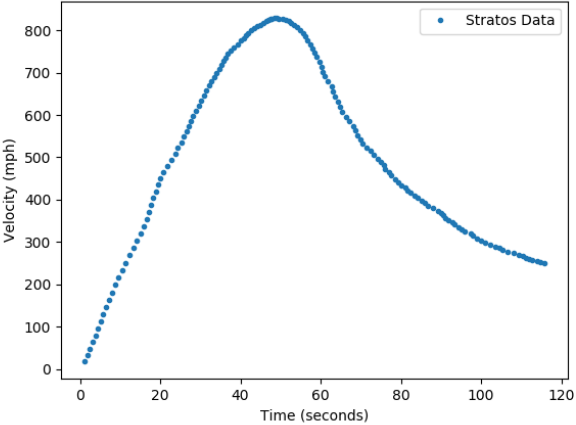

Figure: Velocity graph of the Stratos jump.*

*Data extracted from graphics in Stratos Summit Report

First velocity model

$$\frac{dv}{dt} = g, \quad \text{with} \quad v(0) = 0.$$

where

- $g = 9.81\text{m/s}^2$ is the gravitational acceleration

The solution of this first-order initial value problem is:

$$v(t) = gt.$$

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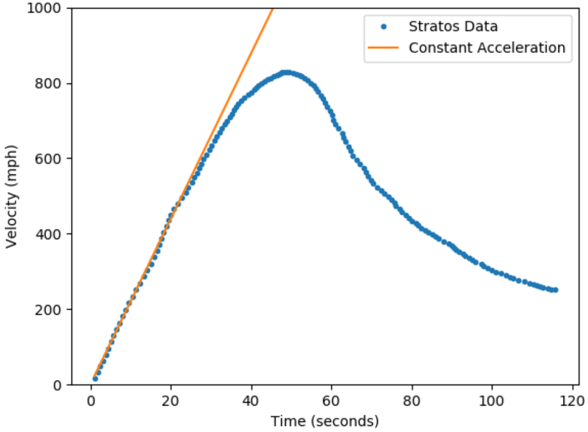

Figure: Comparing model and observations

How did we do?

- As we see, models approximate reality, but have errors!
- The model is very close during the first 20 seconds
- After this, the true velocity starts to differ

Question

Can we account for the differences and make a better model?

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Figure: What forces are in play as Felix falls?

Introduce a drag force

- Linear drag:

$$F_{\text{drag}} = -\gamma m v.$$

- Differential equation:

$$\frac{dv}{dt} = g - \gamma v, \quad \text{with} \quad v(0) = 0.$$

- This is a separable equation!
- Solution

$$v(t) = \frac{g}{\gamma}\left(1 - e^{-\gamma t}\right).$$

- $\gamma = g/v_{\text{term}}$

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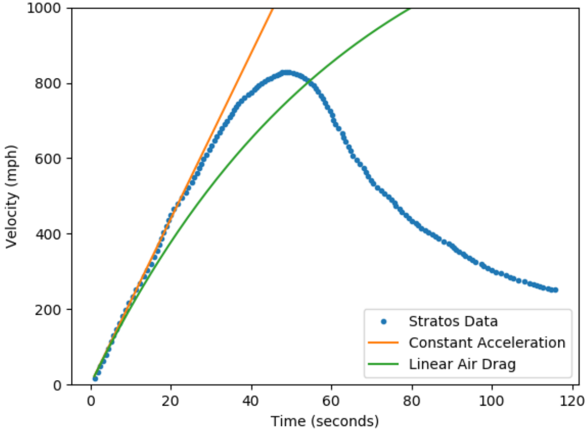

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Can we explain why our new model is worse?

- Physically, we had the wrong drag force...
- At high speeds, drag behaves *quadratically*

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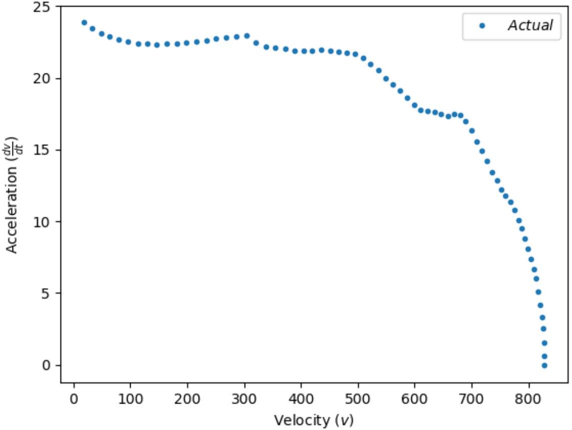

Figure: Graph of dv/dt versus v

Acceleration versus velocity fit

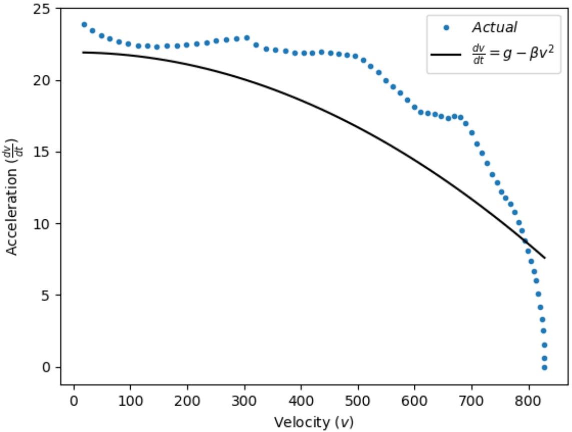

Figure: $\beta = 2.25 \cdot 10^{-5}$

Introduce a drag force

- New differential equation:

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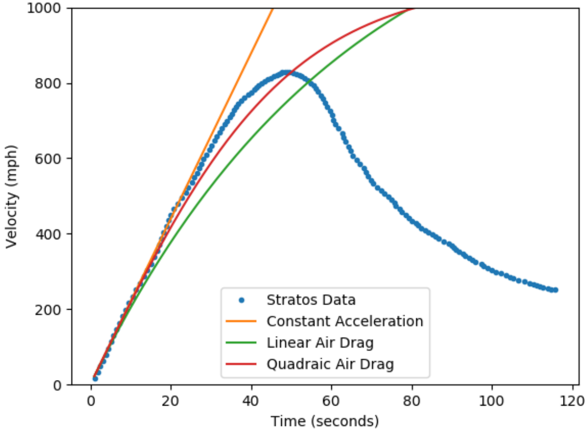

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- The new model is much better!
- Still not super great for later times...
- What are we not taking into acount?
- Drag is proportional to density!

$$F_{\text{drag}} = -\text{constant} \cdot mv^2\rho.$$

- Density varies with elevation

$$\rho = \rho_0 e^{-h/\lambda}.$$

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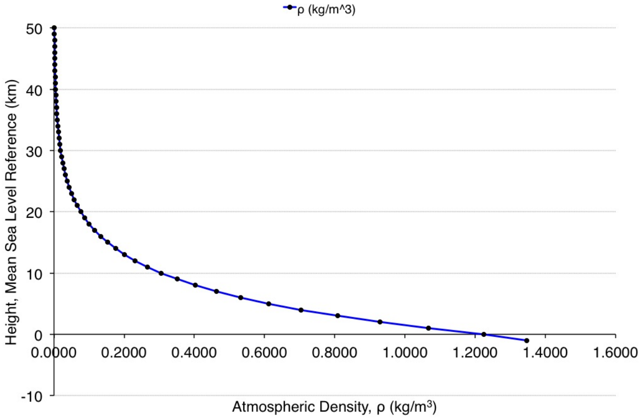

Figure: Density is very near 38 km where Felix starts.

Updated Drag Forcing

- During the initial time $v(t) \approx gt$
- Therefore height is

$$h(t) \approx h_0 - \frac{1}{2}gt^2, \quad h_0 = 24.21 \text{ miles}.$$

- Consequently,

$$h \approx h_0 - \frac{v^2}{2g}.$$

- Updated drag:

$$\begin{aligned} F_{\text{drag}} &= -\text{constant} \cdot mv^2 \rho_0 \exp(h/\lambda) \\ &= -mv^2 \exp(\alpha + \beta v^2). \end{aligned}$$

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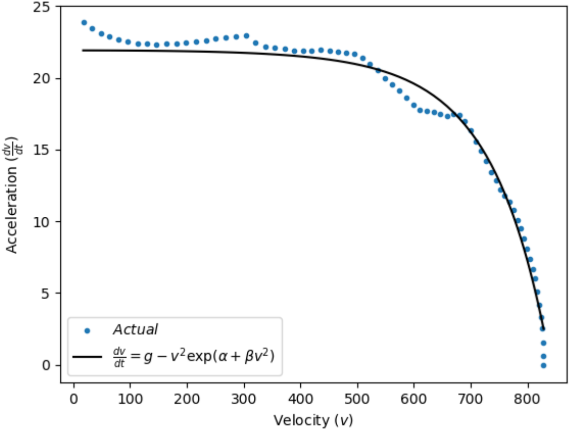

Figure: $\alpha = -13.59$, $\beta = 4.54 \cdot 10^{-6}$

Generic fitting

- New and improved model:

$$\frac{dv}{dt} = g - v^2 \exp(\alpha + \beta v), \quad \text{with} \quad v(0) = 0.$$

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- To solve exactly, we'd need to integrate

$$\int \frac{1}{g - v^2 e^{\alpha + \beta v}} dv.$$

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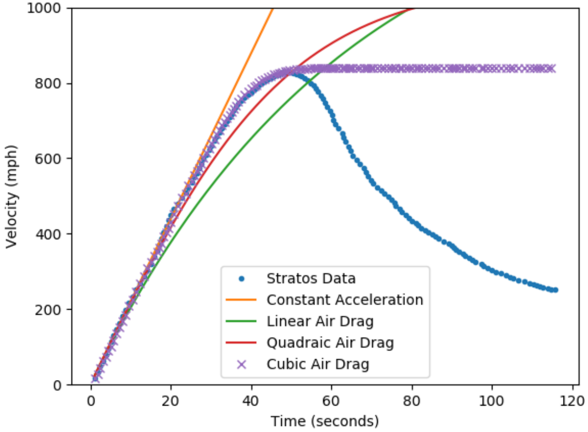

Figure: Comparing model and observations

Thank you!

Figure: Sticking the landing with modelling using first order ODEs!