Representation Theory and the Matrix Bochner Problem

GAP XVI

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June 2, 2018

Outline

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Spherical function definition

DATA:

- real, semisimple Lie group G
- compact subgroup K with $U(\mathfrak{g}^{\mathbb{C}})^K$ commutative
- Cartan subgroup A which is 1-dim
- compact centralizer $M = C_G(A) \cap K$
- unitary representation $\tau : K \times K \to \mathrm{End}_{\mathbb{C}}(E)$

Definition

$F : G \to E$ is τ-**spherical** if smooth and

$$F(k_1 a k_2^{-1}) = \tau(k_1, k_2) F(a) \; \forall k_1, k_2 \in K, \; a \in A.$$

Examples

- unitary rep $\widetilde{\pi} : G \to \widetilde{E} = \mathrm{End}_{\mathbb{C}}(\widetilde{V})$ is $\widetilde{\tau}$ spherical for

$$\widetilde{\tau} : K \times K \to \mathrm{End}_{\mathbb{C}}(\widetilde{E}), \ \ \widetilde{\tau}(k_1, k_2) : \varphi \mapsto \widetilde{\pi}(k_1)\varphi\widetilde{\pi}(k_2).$$

More generally, consider $V \subseteq \widetilde{V}$ $\widetilde{\pi}|_K$-inv, and $\pi : K \to E = \mathrm{End}(V)$ the corresp. rep

- $F_\pi^{\widetilde{\pi}}(g) := P_V \circ \pi(g) : G \to E$ is τ-spherical for

$$\tau : K \times K \to \mathrm{End}_{\mathbb{C}}(E), \ \ \tau(k_1, k_2) : \varphi \mapsto \pi(k_1)\varphi\pi(k_2).$$

  ie.

$$F_\pi^{\widetilde{\pi}}(k_1 a k_2^{-1}) = \pi(k_1) F_\pi^{\widetilde{\pi}}(a)\pi(k_2).$$

Spherical function properties

Take $F : G \to E$ to be τ-spherical:

- $F|_A$ determines F (by global Cartan decomp $G = KAK$)
- $F|_A$ is M-equivariant

$$F(A) \subseteq E^M := \{v \in E : \tau(m, m)v = v\}.$$

- $X \cdot F$ is τ-spherical for all $X \in (\mathfrak{g}^{\mathbb{C}})^K$
- Casselman-Miličić map:

$$\Pi_\tau : U(\mathfrak{g}^{\mathbb{C}})^K \to \mathrm{End}_{\mathbb{C}}(E^M) \otimes_{\mathbb{C}} \mathcal{D}(A).$$

$$(X \cdot F)_A = \Pi_\tau(X) \cdot F_A$$

Irreducible spherical functions

Definition

if $\pi : K \to V, \widetilde{\pi} : G \to \widetilde{V}$ are both irreducible, unitary then $F_{\pi}^{\widetilde{\pi}} : G \to E = \mathrm{End}(V)$ is called an **irreducible spherical function**

- Orthogonality ($\widetilde{\pi}_1 \neq \widetilde{\pi}_2$ irred. unitary):

$$\int_A F_{\widetilde{\pi}_1}^{\pi}(a)^* F_{\widetilde{\pi}_2}^{\pi}(a) da = 0$$

- Eigenfunctions of every $C \in U(\mathfrak{g}^{\mathbb{C}})^K$ (Casselman-Miličić):

$$\Pi_{\tau}(C) \cdot F_{\widetilde{\pi}}^{\pi}|_A = \lambda(\widetilde{\pi}, C) F_{\widetilde{\pi}}^{\pi}|_A.$$

Outline

Matrix Polynomial Construction

- Dual object

$$\widehat{G} = \{\text{equiv. classes of irred. unitary reps of } G\}.$$

- Refinement

$$\widehat{G}(\pi) = \{\pi \in \widehat{G} : \pi \text{ occurs in } \widetilde{\pi}|_K \text{ with multiplicity } 1\}.$$

$$\widehat{G}(\pi, n) = \{\pi \in \widehat{G}(\pi) : \deg(\widetilde{\pi}) - \deg(\pi) = n\}.$$

$$T_n = T_{n,\pi} := \bigoplus_{\widetilde{\pi} \in \widehat{G}(\pi,n)} F_{\pi}^{\widetilde{\pi}} : G \to E^{\oplus |\widehat{G}(\pi,n)|}.$$

Matrix Polynomial Construction

- $E^M = \{\varphi \in E : \pi(m), \varphi \text{ commute } \forall m \in M\}$

For sufficiently nice (G, K) (eg. $G = U(n+m), K = U(n) \times U(m)$, or $G = SU(2) \times SU(2), K = \text{diag}$),

$$\ell := |\widehat{G}(\pi, n)| = \dim(E^M).$$

- T_n restricts to an $\ell \times \ell$-matrix valued function on A
- T_0 is invertible away from a few points
- $P_n(x) = T_n(a(x))T_0^{-1}(a(x))$ is a matrix-valued polynomial of degree n for appropriate parametrization $a(x)$ of A

Properties

- $\deg(P_n(x)) = n$ with nonsingular leading coeff for all $n \geq 0$
- Orthogonality:

$$\int P_n(x)W(x)P_m(x)^* dx = 0, \quad W(x) = T_0(a(x))T_0(a(x))^*.$$

- Eigenfunctions of second-order differential operator:

$$P_n(x) \cdot T_0(a(x))\Pi_\tau(C)^* T_0(a(x))^{-1} = \Lambda(n)P_n(x)$$

NOTE: the right action is a convention for compatibility with noncommutative terms in the inner product

Outline

Matrix Bochner Problem

Problem

Classify all orthogonal matrix polynomials which are eigenfunctions of a second-order differential operator.

- Equiv. classify all weights $W(x)$
- More generally, calculate

$$\mathcal{D}(W) = \{D : \exists \Lambda(n) \text{ s.t. } P_n(x) \cdot D = \Lambda(n) P_n(x) \forall n\}.$$

Classical (scalar) examples

- Hermite polynomials corresp. to weight $W(x) = e^{-x^2}$ are eigenfunctions of $\partial_x^2 - 2x$
- Laguerre polynomials corresp. to weight $W(x) = x^b e^{-x} 1_{(0,\infty)}(x)$ are eigenfunctions of $\partial_x^2 x + \partial_x(b + 1 - x)$
- Jacobi polynomials corresp. to weight $W(x) = (1 - x)^a (1 + x)^b 1_{(-1,1)}(x)$ are eigenfunctions of $\partial_x^2 (1 - x)^2 + \partial_x(a - b - (a + b + 2)x)$

Theorem (Bochner)

Up to affine trans. these are all possible cases which are 1×1

Outline

The algebra $\mathcal{D}(W)$

- subalgebra of $M_N(\mathbb{C}[x, \partial_x])^{op}$
- closed under adjoint operation

$$D \mapsto D^\dagger = W(x) D^* W(x)^{-1}.$$

- noncommutative, semiprime PI algebra
- finitely generated algebra over $\mathbb{C}$ (nontrivial!)
- center $\mathcal{Z}(W)$ is finitely generated over $\mathbb{C}$

Local structure of $\mathcal{D}(W)$

- Ring of fractions

$$\mathcal{F}(W) = \{B^{-1}A : A, B \in \mathcal{Z}(W),\ B \text{ not a zero divisor}\}.$$

- $\mathcal{F}_i(W)$, $i = 1, \ldots, r$ fraction field of i'th irred. component of Spec($\mathcal{Z}(W)$)

Theorem (-,Yakimov 2018)

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i(W)).$$

Classification

- $n_1 + \cdots + n_r$ is the **rank** of $\mathcal{D}(W)$ (bounded by ℓ)

Theorem (-,Yakimov 2018)

Let $W(x)$ be an $\ell \times \ell$ weight matrix solving Bochner, with $\mathcal{D}(W)$ having rank ℓ. Then

$$W(x) = U(x) diag(r_1(x), \dots, r_\ell(x)) U(x)^*$$

for some rational matrix $U(x)$ and some classical weights $r_1(x), \dots, r_\ell(x)$ and

$$P_n(x) = diag(p_{1n}(x), \dots, p_{\ell,n}(x)) \cdot L$$

for some differential operator L.

Outline

Weight matrix

- Recall from construction $\pi : K \to E = \mathrm{End}(V)$ gave us $F_{\pi}^{\widetilde{\pi}}$ which gave us a matrix weight $W(x) = W_{\pi}(x)$.
- The above classification says

$$W_{\pi}(x) = U_{\pi}(x)\mathrm{diag}(r_1(x), \ldots, r_{\ell}(x))U_{\pi}(x)$$

- This type factorization was observed previously in special cases, now explained!

Outline

Spherical functions

- $L = L_\pi$ conjugates $\widetilde{C} = T_0(x)\Pi_\tau(C)^* T_0(x)^{-1}$ to diagonal

$$L_\pi \widetilde{C} = \operatorname{diag}(D_1, \dots, D_r) L_\pi$$

  for some differential operators $D_1, \dots, D_R$.
- therefore the images of the Casimir operators under the Π_τ are all related
- the derivatives of τ-spherical functions are also all related

Future work: try to understand the values of $U_\pi(x)$ and L_π from Lie theoretic perspective

Thanks for listening!

- New paper: https://arxiv.org/abs/1803.04405
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