Representation Theory and the Matrix Bochner Problem

ICM 2018 Satellite Cusco

W.R. Casper

Department of Mathematics
Louisiana State University

August 27, 2018

Outline

Outline

Two very different topics

There is a surprising connection between two unlikely topics:

Topic 1: Irreducible unitary representations of Gelfand Pairs

Certain nice pairs of Lie groups (G, K) with $K \subseteq G$ such as

$$(\mathrm{SU}(2n+1), \mathrm{U}(n)),\ (\mathrm{SO}(n), \mathrm{SO}(n-1)),\ (\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{diag}), \dots$$

Topic 2: Orthogonal Matrix Polynomials satisfying 2nd order matrix-valued differential equations

matrix-valued generalizations of the classical orthogonal polynomials of Hermite, Laguerre, and Jacobi.

Our topic

- **One direction:** we can use the representation theory to construct examples of polynomials
- **Opposite direction:** the polynomials describe the analytic behavior of spherical functions on each Gelfand pair —— spherical functions in turn describe the behavior of the matrix entries of irreducible unitary representations
- **New result:** classification of the orthogonal matrix polynomials satisfying 2^{nd} order equations (ie. matrix Bochner problem)
- We will discuss this classification and how it might fit into the above relationship

Outline

Multiplicity-free representations

For the puroposes of this talk

- G will be a real, compact, connected semisimple Lie group
- K will be a closed subgroup
- $\widehat{G}, \widehat{K}$ dual objects

An irreducible unitary representation $\pi : G \to \mathrm{End}(V_\pi)$ restricts to

$$\pi\big|_K \cong \bigoplus_{[\tau]\in\widehat{K}} \tau^{\oplus m_\pi([\tau])}.$$

Definition

An equivalence class $[\tau] \in \widehat{K}$ is **multiplicity free** if $m_\pi(\tau) \leq 1$ for all $[\pi] \in \widehat{G}$.

Spherical function definition

Let $[\pi] \in \widehat{G}$ and $[\tau] \in \widehat{K}$ be multiplicity free with $m_\pi(\tau) = 1$ and $b : V_\tau \to V_\pi$ the corresponding unitary K-equivariant embedding with Hermitian adjoint $b^* : V_\pi \to V_\tau$. Then

$$F_\pi^\tau : G \to \mathrm{End}(V_\tau), \quad F_\pi^\tau(g) \mapsto b^*\pi(g)b$$

satisfies the property that

$$F(k_1 g k_2) = \tau(k_1)F(g)\tau(k_2), \ \forall k_1, k_2 \in K, \ g \in G. \qquad (1)$$

Definition

A function $F : G \to \mathrm{End}(V_\tau)$ satisfying Equation 1 is called a τ**-spherical function.**

Elementary spherical functions

The space $C_\tau(G)$ of τ-spherical functions has an inner product

$$\langle F_1, F_2 \rangle := \int_G \mathrm{Tr}(F_1(g)^* F_2(g)) dg.$$

Definition

A spherical function of the form F_π^τ (as constructed in the previous slide) is called a **elementary spherical function.**

Theorem (van Pruijssen, Roman)

The elementary spherical functions form an orthonormal basis for $C_\tau(G)$.

Outline

Zonal spherical functions

Definition

A **zonal spherical function** is a function in $C_1(G)$, ie. a function

$$f : G \to \mathbb{C}, \quad f(k_1 g k_2) = f(g)\ \forall k_1, k_2 \in K,\ g \in G.$$

If (G, K) is a compact, rank 1 Gelfand pair then $C_1(G)$ is a polynomial ring $C_1(G) = \mathbb{C}[\phi]$. The function ϕ is elementary and is called the **fundamental zonal spherical function.**

We assume moving forward that (G, K) is a compact, rank 1 Gelfand pair.

Degree map

- There exists $B(\tau) \subseteq \widehat{G}$ such that

$$C_\tau(G) = \bigoplus_{[\pi]\in B(\tau)} \mathbb{C}[\phi] F_\pi^\tau.$$

- More specifically for all $[\eta] \in \widehat{G}$ with $m_\eta(\tau) = 1$

$$F_\eta^\tau = \sum_{[\pi]\in B(\tau)} q_{\eta,\pi}(\phi) F_\pi^\tau \quad \text{for some } q_{\eta,\pi}(\phi) \in \mathbb{C}[\phi]$$

Definition

We define the **degree** of $[\eta] \in \widehat{G}$ with $m_\eta(\tau) = 1$ to be

$$\deg([\eta]) = \max_{\pi\in B(\tau)} (\deg q_{\eta,\pi}).$$

Orthogonal polynomial definition

Minor miracle: For each $n \geq 0$ the set

$$B(\tau, n) = \{[\pi] \in \widehat{G} : m_{[\pi]}([\tau]) = 1,\ \deg([\pi]) = n\}$$

has order exactly $\ell := |B(\tau)|$.

Definition

For each $n \geq 0$, fix an ordering of $B(\tau, n)$ and define

$$P_n(\phi) = [q_{\eta,\pi}(\phi) : ([\eta], [\pi]) \in B(\tau, n) \times B(\tau)].$$

$$W(g) := [\mathrm{Tr}(F_\pi(g)^* F_{\pi'}(g)) : ([\pi], [\pi']) \in B(\tau) \times B(\tau)].$$

Orthogonal polynomial properties

- $\deg(P_n(x)) = n$ with nonsingular leading coeff for all $n \geq 0$
- Orthogonality:

$$\int_{-1}^{1} P_n(x)^* W(x) P_m(x)(1-x)^\alpha (1+x)^\beta dx = 0.$$

  for $(1-x)^\alpha(1+x)^\beta$ corresponding to the change of variables $x = c\phi + d$ where c and d are chosen so that $x : G \to [-1, 1]$ surjectively.
- The $P_n(x)$ are all eigenfunctions of a second-order differential operator:

$$(1-x^2)P_n''(x) + (B_1 x + B_0)P_n'(x) + CP_n(x) = P_n(x)\Lambda_n,$$

  where here the $B_1, B_0, C, \Lambda_n \in M_\ell(\mathbb{C})$.

Outline

Matrix Bochner problem

Problem

Classify all orthogonal matrix polynomials which are eigenfunctions of a second-order differential operator.

- Equiv. classify all weights $W(x)$
- More generally, calculate

$$\mathcal{D}(W) = \{D : \exists \Lambda(n) \text{ s.t. } P_n(x) \cdot D = \Lambda(n) P_n(x) \forall n\}.$$

Classical (scalar) examples

- Hermite polynomials corresp. to weight $W(x) = e^{-x^2}$ are eigenfunctions of $\partial_x^2 - 2x$
- Laguerre polynomials corresp. to weight $W(x) = x^b e^{-x} 1_{(0,\infty)}(x)$ are eigenfunctions of $\partial_x^2 x + \partial_x (b + 1 - x)$
- Jacobi polynomials corresp. to weight $W(x) = (1 - x)^a (1 + x)^b 1_{(-1,1)}(x)$ are eigenfunctions of $\partial_x^2 (1 - x)^2 + \partial_x (a - b - (a + b + 2)x)$

Theorem (Bochner)

Up to affine trans. these are all possible cases which are 1×1

Outline

The algebra $\mathcal{D}(W)$

- subalgebra of $M_N(\mathbb{C}[x, \partial_x])^{op}$
- closed under adjoint operation

$$D \mapsto D^\dagger = W(x) D^* W(x)^{-1}.$$

- noncommutative, semiprime PI algebra
- finitely generated algebra over $\mathbb{C}$ (nontrivial!)
- center $\mathcal{Z}(W)$ is finitely generated over $\mathbb{C}$

Local structure of $\mathcal{D}(W)$

- Ring of fractions

$$\mathcal{F}(W) = \{B^{-1}A : A, B \in \mathcal{Z}(W),\ B \text{ not a zero divisor}\}.$$

- $\mathcal{F}_i(W)$, $i = 1, \dots, r$ fraction field of i'th irred. component of Spec($\mathcal{Z}(W)$)

Theorem (-,Yakimov 2018)

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i(W)).$$

Classification

- $n_1 + \cdots + n_r$ is the **rank** of $\mathcal{D}(W)$ (bounded by ℓ)

Theorem (-,Yakimov 2018)

Let $W(x)$ be an $\ell \times \ell$ weight matrix solving Bochner, with $\mathcal{D}(W)$ having rank ℓ. Then

$$W(x) = U(x) diag(r_1(x), \dots, r_\ell(x)) U(x)^*$$

for some rational matrix $U(x)$ and some classical weights $r_1(x), \dots, r_\ell(x)$ and

$$P_n(x) = diag(p_{1n}(x), \dots, p_{\ell,n}(x)) \cdot L$$

for some differential operator L.

Outline

Weight matrix

- Recall that a fixed multiplicity free representation τ of K lead to a matrix weight $W(x) = W_\tau(x)$.
- The above classification says

$$W_\tau(x) = U_\tau(x)\mathrm{diag}(r_1(x), \ldots, r_\ell(x))U_\tau(x)^*$$

- This type factorization was observed previously in special cases, now explained!

Outline

Differential operator construction

- The polynomials of $W_\tau(x)$ are eigenfunctions of a 2nd order differential operator D_τ
- D_τ is (essentially) given by a Casimir element $X \in U(\mathfrak{g}^{\mathbb{C}})^K$
- the action of X preserves τ-spherical functions

$$X \cdot F_\eta^\tau = \sum_{\pi \in B(\tau)} a_{\eta,\pi}(\phi) F_\pi^\eta$$

- induces a map $\Pi_\tau(X) : \mathbb{C}[\phi]^{\oplus \ell} \to \mathbb{C}[\phi]^{\oplus \ell}$
- $\Pi_\tau(X)$ is a matrix differential operator (essentially the image of X under the Casselman-Miličić map)
- D_τ is obtained from Π_τ after conjugation and change of variable

Diagonalization of differential operator

- L_τ diagonalizes D_τ:

$$L_\tau D_\tau = \mathrm{diag}(D_1, \dots, D_\ell) L_\tau$$

  for some differential operators $D_1, \dots, D_\ell$.
- the values of $\Pi_\tau(X)$ are all related for different τ
- induces relationships between τ-spherical and μ-spherical functions for $[\tau], [\mu] \in \widehat{K}$

Future work: try to understand the values of $U_\tau(x)$ and L_τ from Lie theoretic perspective

Thanks for listening!

- New paper: https://arxiv.org/abs/1803.04405

- Bochner, Salomon. Über Sturm-Liouvillesche Polynomsysteme, Mathematische Zeitschrift 1929.
- Kreĭn, M. Infinite *J*-matrices and the matrix-moment problem, DokladyAkad. Nauk SSSR 1949
- Geiger, Joel and Horozov, Emil and Yakimov, Milen. Noncommutative bispectral Darboux transformations, Transactions AMS 2017
- van Pruijssen, Marten and Román, Pablo. Matrix valued classical pairs related to compact Gelfand pairs of rank one, SIGMA 10 2014