The Matrix Bochner Problem

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Outline

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1. Orthogonal matrix polynomials
   - Classical orthogonal polynomials
   - Orthogonal matrix polynomials
2. The Algebra $\mathcal{D}(W)$
   - Algebras of differential operators
   - Consequences

The classical orthogonal polynomials

- Hermite polynomials

$$p_{\text{herm}}(x,n)'' - 2xp'_{\text{herm}}(x,n) = -2np_{\text{herm}}(x,n)$$

$$\int_{-\infty}^{\infty} p_{\text{herm}}(x,m)e^{-x^2}p_{\text{herm}}(x,n)dx = 0 \text{ for } m \neq n$$

$$\begin{aligned} p_{\text{herm}}(x,0) &= 1 \\ p_{\text{herm}}(x,1) &= x \\ p_{\text{herm}}(x,2) &= x^2 - 1 \\ p_{\text{herm}}(x,3) &= x^3 - 3x \\ p_{\text{herm}}(x,4) &= x^4 - 6x^2 + 3 \end{aligned}$$

The classical orthogonal polynomials

- Laguerre polynomials

$$xp_{\text{lag}}(x,n)'' + (b+1-x)p'_{\text{lag}}(x,n) = -np_{\text{lag}}(x,n)$$

$$\int_0^\infty p_{\text{lag}}(x,m)x^b e^{-x} p_{\text{lag}}(x,n)dx = 0 \text{ for } m \neq n$$

$$p_{\text{lag}}(x,0) = 1$$
$$p_{\text{lag}}(x,1) = -x + a + 1$$
$$p_{\text{lag}}(x,2) = \frac{1}{2}(x^2 - (2b+4)x + (b+1)(b+2))$$

The classical orthogonal polynomials

- Jacobi polynomials

$$(1-x^2)p_{\text{jac}}(x,n)'' + (\beta-\alpha+(\beta+\alpha+2)x)p_{\text{jac}}'(x,n) = (-n^2+(\beta+\alpha+1)n)p_{\text{jac}}(x,n)$$

$$\int_{-1}^{1} p_{\text{jac}}(x,m)(1-x)^{\alpha}(1+x)^{\beta}p_{\text{jac}}(x,n)dx = 0 \text{ for } m \neq n$$

$$p_{\text{jac}}(x,0) = 1$$
$$p_{\text{jac}}(x,1) = \frac{\alpha+\beta+2}{2}x - \frac{\beta-\alpha}{2}$$

Bochner's Theorem

Theorem (Bochner 1929)

Up to affine transformation, the only orthogonal polynomials on $\mathbb{R}$ which are eigenfunctions of a second order differential operator are the classical orthogonal polynomials: the Hermite, Laguerre, and Jacobi polynomials.

Generalizations???

- exceptional orthogonal polynomials
- multi-variate versions
- discrete versions (with difference operators)
- **matrix orthogonal polynomials**

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1. Orthogonal matrix polynomials
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   - Orthogonal matrix polynomials

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Matrix orthogonality

Definition

A **weight matrix** is a function $W(x) : \mathbb{R} \to M_N(\mathbb{C})$ which is smooth, positive definite, and Hermitian on an interval (x_0, x_1) and zero outside of (x_0, x_1) and which has finite moments.

A matrix-valued inner product on $N \times N$ matrix-valued polynomials:

$$\langle P(x), Q(x) \rangle_W = \int P(x) W(x) Q(x)^* dx.$$

More generally, we can replace $W(x)dx$ with a wilder matrix-valued measure.

Orthogonal Matrix Polynomials

Definition (Kreĭn 1949)

A sequence of orthogonal matrix polynomials for a weight $W(x)$ is a sequence $P(x,n)$ of $N \times N$ matrix-valued polynomials

- $\deg(P(x,n)) = n$ with nonsingular leading coefficient
- $\langle P(x,m), P(x,n)\rangle_W = 0$ for $m \neq n$

- Polynomials are unique if normalized or monic

Generalization of classical orthogonal polynomials

Question

Are there interesting matrix generalzations of the classical orthogonal polynomials?

- Matrix-valued orthogonal polynomials for a weight $W(x)$
- Eigenfunctions of some second-order differential equation

$$\frac{d^2}{dx^2}P(x,n)A_2(x) + \frac{d}{dx}P(x,n)A_1(x) + P(x,n)A_0(x) = \Lambda(n)P(x,n)$$

- left vs. right multiplication is very important!!

The Matrix Bochner problem

Problem (Matrix Bochner problem)

Find all weight matrices $W(x)$ whose sequences of orthogonal matrix polynomials $P(x,n)$ satisfy a second-order differential equation

$$\frac{d^2}{dx^2}P(x,n)A_2(x)+\frac{d}{dx}P(x,n)A_1(x)+P(x,n)A_0(x)=\Lambda(n)P(x,n)$$

for some matrix-valued functions $A_i(x)$ and matrices $\Lambda(n)$.

In terms of right-acting operators:

$$P(x,n)\cdot\mathfrak{D}=\Lambda(n)P(x,n),\ \ \mathfrak{D}=\partial_x^2A_2(x)+\partial_xA_1(x)+A_0(x).$$

Bochner pairs

- By a result of Grünbaum and Tirao, we can take $\mathfrak{D}$ to be W-symmetric:

$$\langle P(x) \cdot \mathfrak{D}, Q(x) \rangle_W = \langle P(x), Q(x) \cdot \mathfrak{D} \rangle_W.$$

Definition

A **Bochner pair** is a pair $(W(x), \mathfrak{D})$ with $W(x)$ a weight matrix and $\mathfrak{D}$ a W-symmetric second order differential operator.

Problem (Matrix Bochner problem)

Classify all matrix Bochner pairs.

Examples

[Hermite-type:]

$$\mathfrak{D} = \partial_x^2 I + \partial_x \begin{pmatrix} a-2x & 4b(2-a(a+2x)) \\ 0 & -a-2x \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$W(x) = \begin{pmatrix} 4b^2(a+2x)^2 + 16e^{2ax} & 2b(a+2x) \\ 2b(a+2x) & 1 \end{pmatrix} e^{-x^2-ax}$$

Examples

[Laguerre-type:]

$$\mathfrak{D} = \partial_x^2 xI + \partial_x \begin{pmatrix} b+a+2-x & a+2-(a/b)x \\ 0 & b-x \end{pmatrix} + \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

$$W(x) = \begin{pmatrix} cx^{a+2} + (b-x)^2 & -b(b-x) \\ -b(b-x) & b^2 \end{pmatrix} x^{b-1} e^{-x}.$$

Examples

[Jacobi-type:]

$$\alpha = d(-b^2c^2 + b^2 + 1 + bc(b^2c^2 + b^2 - 1))/2 - 1$$
$$\beta = d(-b^2c^2 + b^2 + 1 - bc(b^2c^2 + b^2 - 1))/2 - 1$$

$$\mathfrak{D} = \partial_x^2(1 - x^2)I - \partial_x x(\alpha + \beta + 4)I$$
$$+ \partial_x \begin{pmatrix} x(\beta - \alpha)d - 2bc & -2b \\ 2bc^2 - 2/b & x(\beta - \alpha)d + 2bc) \end{pmatrix}$$
$$+ \frac{d}{2}(b^2c^2 + b^2 - 1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$W(x) = (1-x)^{\alpha}(1+x)^{\beta}\begin{pmatrix} b^2 + (x - bc)^2 & (\beta - \alpha)/b - \frac{\alpha+\beta+2}{bd}x \\ (\beta - \alpha)/b - \frac{\alpha+\beta+2}{bd}x & b^2c^4 - 2c^2 + 1/b^2 + (x + bc)^2 \end{pmatrix}$$

Examples

[Jacobi-type:]

$$\alpha = a - 1 - a^2b^2c/2$$
$$\beta = c - 1 + a^2b^2c/2$$

$$\mathfrak{D} = \partial_x^2(1-x^2)I - \partial_x x \begin{pmatrix} \alpha+\beta+4 & -bc \\ 0 & \alpha+\beta+3 \end{pmatrix}$$
$$+ \partial_x \begin{pmatrix} \beta-\alpha-ab^2c+2 & ab^3c^2-3bc \\ -ab & \beta-\alpha+ab^2c-1 \end{pmatrix} - \frac{a}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$W(x) = (1-x)^\alpha(1+x)^\beta \begin{pmatrix} (\beta-\alpha-a)b^2c-(\beta+\alpha+2+a)cb^2x+(x+1)^2 & b(\beta-\alpha-(\alpha+\beta+2)x) \\ b(\beta-\alpha-(\alpha+\beta+2)x) & a^2b^2+1-x \end{pmatrix}.$$

New phenomena

- **cone** of weights

$$\text{Cone}(\mathfrak{D}) = \{W(x) : (W(x), \mathfrak{D}) \text{ is a Bochner pair}\}.$$

- **algebra** of operators

$$\mathcal{D}(W) = \{\mathfrak{D} : \exists\ \Lambda(n) \text{ with } P(x,n)\cdot\mathfrak{D} = \Lambda(n)P(x,n)\}.$$

- in scalar case $\mathcal{D}(r) = \mathbb{C}[\mathfrak{d}]$
- in the matrix case, $\mathcal{D}(W)$ can have interesting noncommutative structure!!

Example

- Consider the weight matrix

$$W(x) = e^{-x^2}\begin{pmatrix} 1 + a^2x^2 & ax \\ ax & 1 \end{pmatrix}.$$

- $\mathcal{D}(W)$ is generated by four noncommuting operators

$$\mathfrak{D}_1 = \partial_x^2 I + \partial_x \begin{pmatrix} -2x & a \\ 0 & -2x \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathfrak{D}_2 = \partial_x^2 \begin{pmatrix} -a^2/4 & a^3x/4 \\ 0 & 0 \end{pmatrix} + \partial_x \begin{pmatrix} 0 & a/2 \\ -a/2 & a^2x/2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathfrak{D}_3 = \partial_x^2 \begin{pmatrix} -a^2x/2 & a^3x^2/2 \\ -a/2 & a^2x/2 \end{pmatrix} + \partial_x \begin{pmatrix} -(a^2+1) & a(a^2+2) \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a+2/a \\ 0 & 0 \end{pmatrix}$$

$$\mathfrak{D}_4 = \partial_x^2 \begin{pmatrix} -a^3x/4 & a^2(a^2x^2-1)/4 \\ -a^2/4 & a^3x/4 \end{pmatrix} + \partial_x \begin{pmatrix} -a^3/2 & a^2(a^2+2)x/2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a^2/2+1 \\ 1 & 0 \end{pmatrix}$$

Outline

Algebras determine operators!

Consider an algebra of differential operators $\mathcal{A}$ with

1. $\mathcal{A}$ commutative
2. $\mathcal{A}$ contains a Schrödinger operator

$$\partial_x^2 + u(x)$$

Theorem

If $\mathcal{A}$ contains an operator of order 3 then u satisfies the stationary KdV equation

$$\frac{1}{2}u'''(x) = 6uu'(x).$$

Krichever correspondence

Consider an algebra of differential operators $\mathcal{A}$ with

1. $\mathcal{A}$ commutative
2. $\mathcal{A}$ contains operators of order m and n with $\gcd(m, n) = 1$

$\mathcal{A}$	$\longleftrightarrow$	algebraic curve $\mathcal{C}$ with line bundle $\mathcal{L}$
$\mathfrak{d} \in \mathcal{A}$	$\longleftrightarrow$	$p \in \mathcal{C}$
(dual of) kernel of $\mathfrak{d}$	$\longleftrightarrow$	stalk of $\mathcal{L}$ over p
isospectral deformations	$\longleftrightarrow$	jacobian of $\mathcal{C}$

Problems in the matrix case

- $\mathcal{D}(W)$ is noncommutative!
- how do we study $\mathcal{D}(W)$ geometrically?

Theorem (Casper-Yakimov)

The algebra $\mathcal{D}(W)$ is finite as a module over its center $\mathcal{Z}(W)$ and $\mathcal{Z}(W)$ is Noetherian

this requires some tough technology to prove

- Idea: study the *generic* structure of $\mathcal{D}(W)$ over $\mathcal{Z}(W)$
- What does $\mathcal{D}(W)$ look like *locally*?

Generic structure

Theorem (Posner)

A prime PI algebra is generically a central simple algebra over its center.

- our algebra $\mathcal{D}(W)$ is a PI algebra (embeds into a matrix ring)
- unfortunately it is not prime
- it is semiprime and Krull dimension 1

Theorem (Casper-Yakimov)

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i(W)).$$

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Noncommutative bispectral Darboux transformations

- $W(x) \mapsto \widetilde{W}(x)$
- $P(x,n) \mapsto \widetilde{P}(x,n)$

$$\widetilde{P}(x,n) = C(n)^{-1}P(x,n)\cdot\mathfrak{U} \text{ and } P(x,n) = \widetilde{C}(n)^{-1}\widetilde{P}(x,n)\cdot\widetilde{\mathfrak{U}}$$

$$P(x,n)\cdot(\mathfrak{U}\widetilde{\mathfrak{U}}) = C(n)\widetilde{C}(n)P(x,n).$$

$$P(x,n)\cdot(\widetilde{\mathfrak{U}}\mathfrak{U}) = \widetilde{C}(n)C(n)P(x,n).$$

Definition

$\widetilde{W}(x)$ is a noncomm. bispectral Darboux trans. of $W(x)$

Full weights

Definition

The **module rank** of $\mathcal{D}(W)$ is $n_1 + n_2 + \cdots + n_r$ from the previous theorem. If the rank is N, we say that $W(x)$ is **full**.

Theorem (Casper-Yakimov)

If $W(x)$ is full, then $W(x)$ is a noncommutative bispectral Darboux transformation of a direct sum of classical weights.

$$W(x) = T(x) diag(r_1(x), r_2(x), \ldots, r_n(x)) T(x)^*.$$

$$C(n)P(x,n) = diag(p_1(x,n), p_2(x,n), \ldots, p_N(x,n)) \cdot \mathfrak{U}.$$

Sketch of proof

- fullness means we can choose nonzero $\mathfrak{V}_1, \dots, \mathfrak{V}_N \in \mathcal{D}(W)$ with

$$\mathfrak{V}_i \mathfrak{V}_j = 0, \quad i \neq j.$$

- can take the $\mathfrak{V}_i$ to be W-symmetric
- define modules

$$\mathcal{M}_i = \{\vec{\mathfrak{w}} \in \Omega(x)^{\oplus N} : \vec{\mathfrak{w}}^T \mathfrak{V}_j = \vec{0}^T \ \forall j \neq i\}.$$

- $\Omega(x)$, the algebra of differential operators with rational coefficients, is a noncommutative PID:

$$\mathcal{M}_i = \Omega(x)\vec{\mathfrak{u}}_i$$

Sketch of proof

- using $\mathcal{M}_i$, define a matrix differential operator

$$\mathfrak{U} = [\vec{\mathfrak{u}_1}\ \vec{\mathfrak{u}_2}\ \dots\ \vec{\mathfrak{u}_N}]^T, \quad \vec{\mathfrak{u}_i} = \sum_{j=0}^{\ell_i} \partial_x^j \vec{u}_{ji}(x)$$

$$U(x) = [\vec{u}_{\ell_1 1}(x)\ \vec{u}_{\ell_2 2}(x)\ \dots\ \vec{u}_{\ell_N N}(x)]^T$$

- Then

$$R(x) := U(x)W(x)U(x)^* = \operatorname{diag}(r_1(x), \dots, r_N(x)) \text{ is diagonal.}$$

$$\mathfrak{U}W(x)\mathfrak{U}^* R(x)^{-1} = \operatorname{diag}(\mathfrak{d}_1, \mathfrak{d}_2, \dots, \mathfrak{d}_N).$$

Sketch of proof

- $p_i(x, n)$ the sequence of orthogonal polys for $r_i(x)$
- then sequence of matrix-valued functions

$$P(x, n) = \text{diag}(p_1(x, n), p_2(x, n), \ldots, p_N(x, n)) \cdot \mathfrak{U}$$

- satisfies

$$P(x, n) \cdot W(x)\mathfrak{U}^* R(x)^{-1}\mathfrak{U} = \text{diag}(\lambda_1(n), \ldots, \lambda_N(n))P(x, n).$$

$$\int P(x, m)W(x)P(x, n)^* dx = 0, \quad m \neq n.$$

Example

- Consider the weight matrix

$$W(x) = e^{-x^2}\begin{pmatrix} 1 + a^2x^2 & ax \\ ax & 1 \end{pmatrix}.$$

- $\mathcal{D}(W)$ contains

$$\mathfrak{D}_1 = \partial_x^2 I + \partial_x \begin{pmatrix} -2x & a \\ 0 & -2x \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathfrak{D}_2 = \partial_x^2 \begin{pmatrix} -a^2/4 & a^3x/4 \\ 0 & 0 \end{pmatrix} + \partial_x \begin{pmatrix} 0 & a/2 \\ -a/2 & a^2x/2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathfrak{D}_3 = \partial_x^2 \begin{pmatrix} -a^2x/2 & a^3x^2/2 \\ -a/2 & a^2x/2 \end{pmatrix} + \partial_x \begin{pmatrix} -(a^2+1) & a(a^2+2) \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a+2/a \\ 0 & 0 \end{pmatrix}$$

$$\mathfrak{D}_4 = \partial_x^2 \begin{pmatrix} -a^3x/4 & a^2(a^2x^2-1)/4 \\ -a^2/4 & a^3x/4 \end{pmatrix} + \partial_x \begin{pmatrix} -a^3/2 & a^2(a^2+2)x/2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a^2/2+1 \\ 1 & 0 \end{pmatrix}$$

Example

- we have $\mathfrak{V}_1\mathfrak{V}_2 = 0$ for

$$\mathfrak{V}_1 = \mathfrak{D}_2,\quad \mathfrak{V}_2 = a^2\mathfrak{D}_1 + 4\mathfrak{D}_2 - 4I$$

- the modules are

$$\mathcal{M}_1 = \Omega(x)\begin{pmatrix} \partial_x a/2 \\ -\partial_x a^2 x/2 - 1 \end{pmatrix},\quad \mathcal{M}_2 = \Omega(x)\begin{pmatrix} -1 \\ \partial_x a/2 \end{pmatrix}$$

- therefore

$$\mathfrak{U} = \begin{pmatrix} \partial_x a/2 & -\partial_x a^2 x/2 - 1 \\ -1 & \partial_x a/2 \end{pmatrix},\quad U(x) = \begin{pmatrix} a/2 & -a^2 x/2 \\ 0 & a/2 \end{pmatrix}.$$

Example

- the weight satisfies

$$R(x) = U(x)W(x)U(x)^* = (a^2/4)e^{-x^2}I$$

- so $W(x)$ is a noncommutative bispectral Darboux transformation of the Hermite weight
- orthogonal matrix polynomials for $W(x)$ are given in terms of Hermite polynomials by

$$P(x,n) := p_{\text{herm}}(x,n)I \cdot \mathfrak{U},$$

Thanks for listening!

- New paper: https://arxiv.org/abs/1803.04405

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