The Matrix Bochner Problem and Representation Theory

SE Lie Theory Workshop XI

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Outline

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Spherical Zonal Functions

- G locally compact unimodular group
- K compact subgroup
- Haar measure dg normalized by $\int_K dk = 1$

Definition

A **spherical zonal function** on G is $\phi : G \to \mathbb{C}$ satisfying $\phi(e) = 1$ and

$$\phi(x)\phi(y) = \int_K \phi(xky)dk.$$

Spherical Functions

Natural generalization:

- V a finite dimensional vector space
- $[\pi] \in \widehat{K}$ with character χ_π

Definition

A **spherical function of type** $[\pi]$ on G is $\Phi : G \to \mathrm{End}(V)$ satisfying $\Phi(e) = I$ and

$$\Phi(x)\Phi(y) = \int_K \chi_\pi(k^{-1})\Phi(xky)dk.$$

Spherical Function Properties

- Φ a spherical function of type $[\pi]$
- $D \in U(\mathfrak{g})^K$ a differential operator on G which is left G-invariant and right K-invariant

Properties:

(a)

$$\Phi(kgk') = \Phi(k)\Phi(g)\Phi(k')$$

(b) $\Phi|_K$ is a representation of K equivalent to n copies of $[\pi]$

(c) **eigenfunction of a differential operator**:

$$(D \cdot \Phi)(g) = \Phi(g)\Lambda \ \text{ for } \ \Lambda = (D \cdot \Phi)(0)$$

Irreducible Spherical Functions

- $\widetilde{\pi} : G \to \text{End}(\widetilde{V})$ a representation of G
- $\pi : K \to \text{End}(V)$ an irreducible subrepresentation of $\widetilde{\pi}|_K$

Observation:

$$\Phi_\pi^{\widetilde{\pi}}(g) := \int_K \chi_\pi(k^{-1})\widetilde{\pi}(kg)|_V dk \text{ is spherical of type } \pi$$

Definition

The **spherical function of type** π is called irreducible if it is of the above form with $\Phi_\pi^{\widetilde{\pi}}$ for some irreducible π

Orthogonal Basis

- $\widetilde{\pi}_1, \widetilde{\pi}_2 : G \to \mathrm{End}(\widetilde{V})$ irreducible representations of G
- $\pi : K \to \mathrm{End}(V)$ an irreducible subrepresentation of both $\widetilde{\pi}_1|_K$ and $\widetilde{\pi}_2|_K$

Orthogonal basis:

$$\int_G \Phi_{\pi}^{\widetilde{\pi}_1}(g)^* \Phi_{\pi}^{\widetilde{\pi}_2}(g) dg = 0I \text{ iff } [\pi_1] \neq [\pi_2].$$

Proposition

The irreducible spherical functions span the vector space

$$C_{[\pi]}(G) = \{\Phi : G \to \mathrm{End}(V) | \Phi \textit{ is spherical of type } [\pi]\}.$$

Outline

Weight Matrix

Definition

A **weight matrix** is a function $W(x) : \mathbb{R} \to M_N(\mathbb{C})$ which is smooth, positive definite, and Hermitian on an interval (x_0, x_1) and zero outside of (x_0, x_1) and which has finite moments.

A matrix-valued inner product on $N \times N$ matrix-valued polynomials:

$$\langle P(x), Q(x) \rangle_W = \int P(x)W(x)Q(x)^* dx.$$

More generally, we can replace $W(x)dx$ with a wilder matrix-valued measure.

Orthogonal Matrix Polynomials

Definition (Kreĭn 1949)

A sequence of orthogonal matrix polynomials for a weight $W(x)$ is a sequence $P(x,n)$ of $N \times N$ matrix-valued polynomials

- $\deg(P(x,n)) = n$ with nonsingular leading coefficient
- $\langle P(x,m), P(x,n)\rangle_W = 0$ for $m \neq n$

- polynomials are unique if normalized or monic
- the spherical functions can be encoded as orthogonal matrix polynomials

Spherical Functions to Matrix Polynomials

- Algebra of zonal spherical functions = $\mathbb{C}[\phi]$
- ϕ is the **fundamental zonal spherical function**

Can choose $[\widetilde{\pi}_1], \ldots, [\widetilde{\pi}_\ell] \in \widehat{G}$ such that

- $$C_{[\pi]}(G) = \mathbb{C}[\phi]\Phi_\pi^{\widetilde{\pi}_1} \oplus \cdots \oplus \mathbb{C}[\phi]\Phi_\pi^{\widetilde{\pi}_\ell}$$
- For each $n \geq 0$ there exists exactly ℓ classes $[\widetilde{\pi}_{n,1}], \ldots, [\widetilde{\pi}_{n,\ell}]$ such that

$$\Phi_\pi^{\widetilde{\pi}_{n,j}} = \sum_{i=1}^{\ell} p_{n,j,i}(\phi)\Phi_\pi^{\widetilde{\pi}_i}, \quad \max_{ij} \deg p_{n,j,i}(\phi) = n.$$

Spherical Functions to Matrix Polynomials

- Define $P(\phi, n) \in M_\ell(\mathbb{C}[\phi])$ to be the $\ell \times \ell$ matrix with i, j'th entry $p_{n,j,i}(\phi)$
- Cartan decomposition: $G = KAK$, $S := \phi(A) \subseteq \mathbb{R}$, $x = \phi^{-1} : S \to A$.

Orthogonality:

$$\int_S P(x, m)^* W(x) P(x, n) dx = 0I$$

- $W(x)$ and $\ell \times \ell$ matrix expressible in terms of $\Phi_\pi^{\widetilde{\pi}_1}, \ldots, \Phi_\pi^{\widetilde{\pi}_\ell}$ and the derivative of $\phi|_A$.

Outline

Differential Equation

- Since the spherical functions are eigenfunctions of a differential operator, the polynomials $P(x,n)$ are too
- the action of $U(\mathfrak{g})^K$ on $C_{[\pi]}(G)$ restricts to the action of a matrix-valued ordinary differential operator on A
- change of variable gives a matrix-valued differential operator L on $S \subseteq \mathbb{R}$
- The polynomials $P(x,n)$ are eigenfunctions of D with matrix-valued eigenvalues:

$$L \cdot P(x,n) = P(x,n)\Lambda(n), \quad \Lambda(n) \in M_\ell(\mathbb{C}).$$

Matrix Bochner Problem

Problem

Classify all orthogonal matrix polynomials which are eigenfunctions of a second-order differential operator.

- Equiv. classify all weights $W(x)$
- More generally, calculate

$$\mathcal{D}(W) = \{D : \exists \Lambda(n) \text{ s.t. } P_n(x) \cdot D = \Lambda(n) P_n(x) \forall n\}.$$

Examples

[Hermite:]

$$d\mu_{\text{herm}}(x) = e^{-x^2}dx$$

$$\begin{aligned} p_{\text{herm}}(x,0) &= 1 \\ p_{\text{herm}}(x,1) &= x \\ p_{\text{herm}}(x,2) &= x^2 - 1 \\ p_{\text{herm}}(x,3) &= x^3 - 3x \\ p_{\text{herm}}(x,4) &= x^4 - 6x^2 + 3 \end{aligned}$$

Three term recurrence relation

$$xp_{\text{herm}}(x,n) = (1/2)p_{\text{herm}}(x,n+1) + (1/2)p_{\text{herm}}(x,n-1)$$

Examples

[Laguerre:]

$$d\mu_{\text{lag}}(x) = x^b e^{-x} 1_{(0,\infty)}(x)dx$$

$$p_{\text{lag}}(x,0) = 1$$
$$p_{\text{lag}}(x,1) = -x + a + 1$$
$$p_{\text{lag}}(x,2) = \frac{1}{2}(x^2 - (2a+4)x + (a+1)(a+2))$$
$$p_{\text{lag}}(x,3) = \frac{1}{6}(-x^3 + (a+3)(3x^2 - 3(a+2)x + (a+1)(a+2)))$$

Three term recurrence relation

$$xp_{\text{lag}}(x,n) = -(n+1)p_{\text{lag}}(x,n+1) + (2n+1+a)p_{\text{lag}}(x,n) - (n+a)p_{\text{lag}}(x,n$$

Examples

[Jacobi:]

$$d\mu_{\text{jac}}(x) = (1-x)^a(1+x)^b 1_{(-1,1)}(x)dx$$

$$p_{\text{jac}}(x,0) = 1$$
$$p_{\text{jac}}(x,1) = \frac{a+b+2}{2}x - \frac{b-a}{2}$$

Three term recurrence relation

$$xp_{\text{jac}}(x,n) = \frac{2(n+1)(n+1+a+b)}{(2n+a+b+1)(2n+a+b+2)}p_{\text{jac}}(x,n+1) - \frac{(a^2-b^2)}{(2n+a+b+2)(2n+a+b)}p_{\text{jac}}(x,n) + \frac{2(n+a+1)(n+b+1)}{(2n+a+b+1)(2n+a+b)}p_{\text{jac}}(x,n-1)$$

Differential Equations

These examples are special!

- eigenfunctions of a differential operator

[**Hermite:**]

$$[\partial_x^2 - 2x\partial_x] \cdot p_{\text{herm}}(x, n) = -2np_{\text{herm}}(x, n)$$

[**Laguerre:**]

$$[x\partial_x^2 + (b + 1 - x)\partial_x] \cdot p_{\text{lag}}(x, n) = -np_{\text{lag}}(x, n)$$

[**Jacobi:**]

$$[(1-x^2)\partial_x^2+(b-a-(b+a+2)x)\partial_x]\cdot p_{\text{jac}}(x, n) = -n(n+b+a+1)p_{\text{jac}}(x, n)$$

Bochner pairs

- By a result of Grünbaum and Tirao, we can take $\mathfrak{D}$ to be W-symmetric:

$$\langle P(x) \cdot \mathfrak{D}, Q(x) \rangle_W = \langle P(x), Q(x) \cdot \mathfrak{D} \rangle_W.$$

Definition

A **Bochner pair** is a pair $(W(x), \mathfrak{D})$ with $W(x)$ a weight matrix and $\mathfrak{D}$ a W-symmetric second order differential operator.

Problem (Matrix Bochner problem)

Classify all matrix Bochner pairs.

Examples

[Hermite-type:]

$$\mathfrak{D} = \partial_x^2 I + \partial_x \begin{pmatrix} a - 2x & 4b(2 - a(a+2x)) \\ 0 & -a - 2x \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$W(x) = \begin{pmatrix} 4b^2(a+2x)^2 + 16e^{2ax} & 2b(a+2x) \\ 2b(a+2x) & 1 \end{pmatrix} e^{-x^2-ax}$$

Examples

[Laguerre-type:]

$$\mathfrak{D} = \partial_x^2 xI + \partial_x \begin{pmatrix} b+a+2-x & a+2-(a/b)x \\ 0 & b-x \end{pmatrix} + \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

$$W(x) = \begin{pmatrix} cx^{a+2}+(b-x)^2 & -b(b-x) \\ -b(b-x) & b^2 \end{pmatrix} x^{b-1}e^{-x}.$$

Examples

[Jacobi-type:]

$$\alpha = d(-b^2c^2 + b^2 + 1 + bc(b^2c^2 + b^2 - 1))/2 - 1$$
$$\beta = d(-b^2c^2 + b^2 + 1 - bc(b^2c^2 + b^2 - 1))/2 - 1$$

$$\mathfrak{D} = \partial_x^2(1 - x^2)I - \partial_x x(\alpha + \beta + 4)I$$
$$+ \partial_x \begin{pmatrix} x(\beta - \alpha)d - 2bc & -2b \\ 2bc^2 - 2/b & x(\beta - \alpha)d + 2bc) \end{pmatrix}$$
$$+ \frac{d}{2}(b^2c^2 + b^2 - 1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$W(x) = (1 - x)^\alpha (1 + x)^\beta \begin{pmatrix} b^2 + (x - bc)^2 & (\beta - \alpha)/b - \frac{\alpha+\beta+2}{bd}x \\ (\beta - \alpha)/b - \frac{\alpha+\beta+2}{bd}x & b^2c^4 - 2c^2 + 1/b^2 + (x + bc)^2 \end{pmatrix}$$

Examples

[Jacobi-type:]

$$\alpha = a - 1 - a^2b^2c/2$$
$$\beta = c - 1 + a^2b^2c/2$$

$$\mathfrak{D} = \partial_x^2(1-x^2)I - \partial_x x \begin{pmatrix} \alpha+\beta+4 & -bc \\ 0 & \alpha+\beta+3 \end{pmatrix} + \partial_x \begin{pmatrix} \beta-\alpha-ab^2c+2 & ab^3c^2-3bc \\ -ab & \beta-\alpha+ab^2c-1 \end{pmatrix} - \frac{a}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$W(x) = (1-x)^{\alpha}(1+x)^{\beta}\begin{pmatrix} (\beta-\alpha-a)b^2c-(\beta+\alpha+2+a)cb^2x+(x+1)^2 & b(\beta-\alpha-(\alpha+\beta+2)x) \\ b(\beta-\alpha-(\alpha+\beta+2)x) & a^2b^2+1-x \end{pmatrix}.$$

Outline

Big idea

Let $(W, \mathfrak{D})$ be a matrix Bochner pair

- Use the structure of $\mathcal{D}(W)$ over its center $\mathcal{Z}(W)$ to learn about $\mathfrak{D}$
- then use $\mathfrak{D}$ to learn about $W(x)$

Important steps:

1. Show that $\mathcal{Z}(W)$ and $\mathcal{D}(W)$ are affine (nontrivial!)
2. Show that $\mathcal{D}(W)$ is finite over $\mathcal{Z}(W)$
3. Study the *generic* structure of $\mathcal{D}(W)$ over $\mathcal{Z}(W)$
   Rephrased: In a neighborhood of a generic point of $\text{Spec}(\mathcal{Z}(W))$, what does the algebra $\mathcal{D}(W)$ look like?

Operator Adjoints

Theorem (Grünbaum-Tirao)

The algebra $\mathcal{D}(W)$ has a ***adjoint involution****:*

$$\dagger : \mathcal{D}(W) \to \mathcal{D}(W), \quad \mathfrak{D} \mapsto \mathfrak{D}^\dagger$$

$$\langle P(x)\cdot\mathfrak{D}, Q(x)\rangle_W = \langle P(x), Q(x)\cdot\mathfrak{D}^\dagger\rangle_W.$$

- if $W(x)$ is smooth

$$\mathfrak{D}^\dagger = W(x)\mathfrak{D}^* W(x)^{-1}$$

$$\left(\sum_k \partial_x^k A_k(x)\right)^* = \sum_k (-1)^k A_k(x)\partial_x^k.$$

New properties

1. $\mathcal{D}(W)$ is affine (finitely generated)
2. the center $\mathcal{Z}(W)$ of $\mathcal{D}(W)$ is affine
3. the ring $\mathcal{D}(W)$ is module finite over $\mathcal{Z}(W)$
4. $\mathcal{Z}(W)$ is reduced and Krull dimension 1
5. $\mathcal{D}(W)$ is a semiprime PI-algebra of GK-dim 1
6. $\mathcal{D}(W)$ is generically Azumaya over $\mathcal{Z}(W)$

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i(W)).$$

7. the previous isomorphism is involutive

Local structure of $\mathcal{D}(W)$

- Ring of fractions

$$\mathcal{F}(W) = \{B^{-1}A : A, B \in \mathcal{Z}(W),\ B \text{ not a zero divisor}\}.$$

- $\mathcal{F}_i(W)$, $i = 1, \ldots, r$ fraction field of i'th irred. component of Spec($\mathcal{Z}(W)$)

Theorem (-,Yakimov 2018)

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i(W)).$$

Classification

- $n_1 + \cdots + n_r$ is the **rank** of $\mathcal{D}(W)$ (bounded by ℓ)

Theorem (-,Yakimov 2018)

Let $W(x)$ be an $\ell \times \ell$ weight matrix solving Bochner, with $\mathcal{D}(W)$ having rank ℓ. Then

$$W(x) = U(x) diag(r_1(x), \dots, r_\ell(x)) U(x)^*$$

for some rational matrix $U(x)$ and some classical weights $r_1(x), \dots, r_\ell(x)$ and

$$P_n(x) = diag(p_{1n}(x), \dots, p_{\ell,n}(x)) \cdot L$$

for some differential operator L.

Thanks for listening!

- New paper: https://arxiv.org/abs/1803.04405

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