Orthogonal Matrix Polynomials and Representation Theory

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Outline

1. Orthogonal Matrix Polynomials
   - Orthogonal Polynomials
   - Orthogonal Matrix Polynomials
2. The Algebra $\mathcal{D}(W)$
   - Algebras of differential operators
   - Properties of $\mathcal{D}(W)$
   - Consequences

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Orthogonal Polynomials on the Real Line

- $\mu(x)$ a positive measure on $\mathbb{R}$ with finite moments
- inner product on polynomials

$$\langle p(x), q(x)\rangle_\mu := \int p(x)\overline{q(x)}d\mu(x)$$

Definition

A sequence of polynomials $\{p(x,n) : n = 0, 1, 2, \}$ satisfying

- $\deg(p(x,n)) = n$
- $\langle p(x,m), p(x,n)\rangle_\mu = 0$ for $m \neq n$

is a **sequence of orthogonal polynomials** for $\mu(x)$.

- unique if taken to be monic or normalized

Examples

[Hermite:]

$$d\mu_{\text{herm}}(x) = e^{-x^2} dx$$

$$\begin{aligned} p_{\text{herm}}(x,0) &= 1 \\ p_{\text{herm}}(x,1) &= x \\ p_{\text{herm}}(x,2) &= x^2 - 1 \\ p_{\text{herm}}(x,3) &= x^3 - 3x \\ p_{\text{herm}}(x,4) &= x^4 - 6x^2 + 3 \end{aligned}$$

Three term recurrence relation

$$xp_{\text{herm}}(x,n) = (1/2)p_{\text{herm}}(x,n+1) + (1/2)p_{\text{herm}}(x,n-1)$$

Examples

[Laguerre:]

$$d\mu_{\text{lag}}(x) = x^b e^{-x} 1_{(0,\infty)}(x)dx$$

$$p_{\text{lag}}(x,0) = 1$$
$$p_{\text{lag}}(x,1) = -x + a + 1$$
$$p_{\text{lag}}(x,2) = \frac{1}{2}(x^2 - (2a+4)x + (a+1)(a+2))$$
$$p_{\text{lag}}(x,3) = \frac{1}{6}(-x^3 + (a+3)(3x^2 - 3(a+2)x + (a+1)(a+2)))$$

Three term recurrence relation

$$xp_{\text{lag}}(x,n) = -(n+1)p_{\text{lag}}(x,n+1) + (2n+1+a)p_{\text{lag}}(x,n) - (n+a)p_{\text{lag}}(x,n$$

Examples

[Jacobi:]

$$d\mu_{\text{jac}}(x) = (1-x)^a(1+x)^b 1_{(-1,1)}(x)dx$$

$$p_{\text{jac}}(x,0) = 1$$
$$p_{\text{jac}}(x,1) = \frac{a+b+2}{2}x - \frac{b-a}{2}$$

Three term recurrence relation

$$xp_{\text{jac}}(x,n) = \frac{2(n+1)(n+1+a+b)}{(2n+a+b+1)(2n+a+b+2)}p_{\text{jac}}(x,n+1) - \frac{(a^2-b^2)}{(2n+a+b+2)(2n+a+b)}p_{\text{jac}}(x,n) + \frac{2(n+a+1)(n+b+1)}{(2n+a+b+1)(2n+a+b)}p_{\text{jac}}(x,n-1)$$

Differential Equations

These examples are special!

- eigenfunctions of a differential operator

[**Hermite:**]

$$[\partial_x^2 - 2x\partial_x] \cdot p_{\text{herm}}(x, n) = -2np_{\text{herm}}(x, n)$$

[**Laguerre:**]

$$[x\partial_x^2 + (b + 1 - x)\partial_x] \cdot p_{\text{lag}}(x, n) = -np_{\text{lag}}(x, n)$$

[**Jacobi:**]

$$[(1-x^2)\partial_x^2+(b-a-(b+a+2)x)\partial_x]\cdot p_{\text{jac}}(x, n) = -n(n+b+a+1)p_{\text{jac}}(x, n)$$

Applications

- solutions of PDEs
  - separation of variables
  - spectral and pseudo-spectral methods
- polynomial approximation
  - error minimizing approximations
  - root finding
- Hermitian, symmetric, and symplectic random matrices
  - asymptotic behavior of eigenvalues
- representation theory
  - spherical harmonics
  - spherical zonal functions
- quantum mechanics
  - energy levels of the hydrogen atom

Bochner's Theorem

Theorem (Bochner 1929)

Up to affine transformation, the only orthogonal polynomials which are eigenfunctions of a second order differential operator are the classical orthogonal polynomials: the Hermite, Laguerre, and Jacobi polynomials.

Various generalizations of classical orthogonal polynomials:

- exceptional orthogonal polynomials
- multi-variate versions
- orthogonal polynomials satisfying difference equations
- **matrix orthogonal polynomials**

Outline

Weight Matrix

Definition

A **weight matrix** is a function $W(x) : \mathbb{R} \to M_N(\mathbb{C})$ which is smooth, positive definite, and Hermitian on an interval (x_0, x_1) and zero outside of (x_0, x_1) and which has finite moments.

A matrix-valued inner product on $N \times N$ matrix-valued polynomials:

$$\langle P(x), Q(x)\rangle_W = \int P(x)W(x)Q(x)^* dx.$$

More generally, we can replace $W(x)dx$ with a wilder matrix-valued measure.

Orthogonal Matrix Polynomials

Definition (Kreĭn 1949)

A sequence of orthogonal matrix polynomials for a weight $W(x)$ is a sequence $P(x,n)$ of $N \times N$ matrix-valued polynomials

- $\deg(P(x,n)) = n$ with nonsingular leading coefficient
- $\langle P(x,m), P(x,n)\rangle_W = 0$ for $m \neq n$

- Polynomials are unique if normalized or monic
- Some sequences are also eigenfunctions of second-order differential operators

The Matrix Bochner problem

Problem (Matrix Bochner problem)

Find all weight matrices $W(x)$ whose sequences of orthogonal matrix polynomials $P(x,n)$ satisfy a second-order differential equation

$$\frac{d^2}{dx^2}P(x,n)A_2(x)+\frac{d}{dx}P(x,n)A_1(x)+P(x,n)A_0(x)=\Lambda(n)P(x,n)$$

for some matrix-valued functions $A_i(x)$ and matrices $\Lambda(n)$.

In terms of right-acting operators:

$$P(x,n)\cdot\mathfrak{D}=\Lambda(n)P(x,n),\ \ \mathfrak{D}=\partial_x^2A_2(x)+\partial_xA_1(x)+A_0(x).$$

Bochner pairs

- By a result of Grünbaum and Tirao, we can take $\mathfrak{D}$ to be W-symmetric:

$$\langle P(x)\cdot\mathfrak{D}, Q(x)\rangle_W = \langle P(x), Q(x)\cdot\mathfrak{D}\rangle_W.$$

Definition

A **Bochner pair** is a pair $(W(x), \mathfrak{D})$ with $W(x)$ a weight matrix and $\mathfrak{D}$ a W-symmetric second order differential operator.

Problem (Matrix Bochner problem)

Classify all matrix Bochner pairs.

Examples

[Hermite-type:]

$$\mathfrak{D} = \partial_x^2 I + \partial_x \begin{pmatrix} a - 2x & 4b(2 - a(a+2x)) \\ 0 & -a - 2x \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$W(x) = \begin{pmatrix} 4b^2(a+2x)^2 + 16e^{2ax} & 2b(a+2x) \\ 2b(a+2x) & 1 \end{pmatrix} e^{-x^2-ax}$$

Examples

[Laguerre-type:]

$$\mathfrak{D} = \partial_x^2 xI + \partial_x \begin{pmatrix} b+a+2-x & a+2-(a/b)x \\ 0 & b-x \end{pmatrix} + \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

$$W(x) = \begin{pmatrix} cx^{a+2} + (b-x)^2 & -b(b-x) \\ -b(b-x) & b^2 \end{pmatrix} x^{b-1} e^{-x}.$$

Examples

[Jacobi-type:]

$$\alpha = d(-b^2c^2 + b^2 + 1 + bc(b^2c^2 + b^2 - 1))/2 - 1$$
$$\beta = d(-b^2c^2 + b^2 + 1 - bc(b^2c^2 + b^2 - 1))/2 - 1$$

$$\mathfrak{D} = \partial_x^2(1 - x^2)I - \partial_x x(\alpha + \beta + 4)I$$
$$+ \partial_x \begin{pmatrix} x(\beta - \alpha)d - 2bc & -2b \\ 2bc^2 - 2/b & x(\beta - \alpha)d + 2bc) \end{pmatrix}$$
$$+ \frac{d}{2}(b^2c^2 + b^2 - 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$W(x) = (1 - x)^\alpha (1 + x)^\beta \begin{pmatrix} b^2 + (x - bc)^2 & (\beta - \alpha)/b - \frac{\alpha+\beta+2}{bd}x \\ (\beta - \alpha)/b - \frac{\alpha+\beta+2}{bd}x & b^2c^4 - 2c^2 + 1/b^2 + (x + bc)^2 \end{pmatrix}$$

Examples

[Jacobi-type:]

$$\alpha = a - 1 - a^2b^2c/2$$
$$\beta = c - 1 + a^2b^2c/2$$

$$\mathfrak{D} = \partial_x^2(1-x^2)I - \partial_x x \begin{pmatrix} \alpha+\beta+4 & -bc \\ 0 & \alpha+\beta+3 \end{pmatrix} + \partial_x \begin{pmatrix} \beta-\alpha-ab^2c+2 & ab^3c^2-3bc \\ -ab & \beta-\alpha+ab^2c-1 \end{pmatrix} - \frac{a}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$W(x) = (1-x)^\alpha(1+x)^\beta \begin{pmatrix} (\beta-\alpha-a)b^2c-(\beta+\alpha+2+a)cb^2x+(x+1)^2 & b(\beta-\alpha-(\alpha+\beta+2)x) \\ b(\beta-\alpha-(\alpha+\beta+2)x) & a^2b^2+1-x \end{pmatrix}.$$

Applications

- solutions of PDEs
  - noncommutative separation of variables?
- polynomial approximation
  - nonlinear eigenvalue problem
  - matrix-valued special functions
- block-tridiagonal random matrices
  - asymptotic behavior of eigenvalues
- representation theory
  - representations of rank 1 Gelfand pairs
  - spherical functions
- quantum mechanics
  - Dirac equation for a central Coulomb potential

New phenomena

- **cone** of weights

$$\text{Cone}(\mathfrak{D}) = \{W(x) : (W(x), \mathfrak{D}) \text{ is a Bochner pair}\}.$$

- **algebra** of operators

$$\mathcal{D}(W) = \{\mathfrak{D} : \exists\, \Lambda(n) \text{ with } P(x,n)\cdot\mathfrak{D} = \Lambda(n)P(x,n)\}.$$

- in scalar case $\mathcal{D}(r) = \mathbb{C}[\mathfrak{d}]$
- in the matrix case, $\mathcal{D}(W)$ can have interesting noncommutative structure!!

Example

- Consider the weight matrix

$$W(x) = e^{-x^2} \begin{pmatrix} 1 + a^2x^2 & ax \\ ax & 1 \end{pmatrix}.$$

- $\mathcal{D}(W)$ contains

$$\mathfrak{D}_1 = \partial_x^2 I + \partial_x \begin{pmatrix} -2x & a \\ 0 & -2x \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathfrak{D}_2 = \partial_x^2 \begin{pmatrix} -a^2/4 & a^3x/4 \\ 0 & 0 \end{pmatrix} + \partial_x \begin{pmatrix} 0 & a/2 \\ -a/2 & a^2x/2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathfrak{D}_3 = \partial_x^2 \begin{pmatrix} -a^2x/2 & a^3x^2/2 \\ -a/2 & a^2x/2 \end{pmatrix} + \partial_x \begin{pmatrix} -(a^2+1) & a(a^2+2) \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a+2/a \\ 0 & 0 \end{pmatrix}$$

$$\mathfrak{D}_4 = \partial_x^2 \begin{pmatrix} -a^3x/4 & a^2(a^2x^2-1)/4 \\ -a^2/4 & a^3x/4 \end{pmatrix} + \partial_x \begin{pmatrix} -a^3/2 & a^2(a^2+2)x/2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a^2/2+1 \\ 1 & 0 \end{pmatrix}$$

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Algebras determine operators!

Consider an algebra of differential operators $\mathcal{A}$ with

1. $\mathcal{A}$ commutative
2. $\mathcal{A}$ contains a Schrödinger operator

$$\partial_x^2 + u(x)$$

Theorem

If $\mathcal{A}$ contains an operator of order 3 then u satisfies the stationary KdV equation

$$\frac{1}{2}u'''(x) = 6uu'(x).$$

Krichever correspondence

Consider an algebra of differential operators $\mathcal{A}$ with

1. $\mathcal{A}$ commutative
2. $\mathcal{A}$ contains operators of order m and n with $\gcd(m, n) = 1$

$\mathcal{A}$	$\longleftrightarrow$	algebraic curve $\mathcal{C}$ with vector bundle $\mathcal{L}$
$\mathfrak{d} \in \mathcal{A}$	$\longleftrightarrow$	$p \in \mathcal{C}$
(dual of) kernel of $\mathfrak{d}$	$\longleftrightarrow$	stalk of $\mathcal{L}$ over p
isospectral deformations	$\longleftrightarrow$	jacobian of $\mathcal{C}$

Matrix differential operators

Consider an algebra of *matrix* differential operators $\mathcal{A}$ with

- requiring $\mathcal{A}$ to be commutative is too restrictive
- generalization: $\mathcal{A}$ is module finite over its center
- in the scalar case

$$\mathcal{A} \text{ is module finite over its center} \Leftrightarrow \mathcal{A} \text{ is commutative}$$

- matrix case: $\mathcal{A}$ can be noncommutative + finite over center

Big idea

Let $(W, \mathfrak{D})$ be a matrix Bochner pair

- Use the structure of $\mathcal{D}(W)$ over its center $\mathcal{Z}(W)$ to learn about $\mathfrak{D}$
- then use $\mathfrak{D}$ to learn about $W(x)$

Important steps:

1. Show that $\mathcal{Z}(W)$ and $\mathcal{D}(W)$ are affine (nontrivial!)
2. Show that $\mathcal{D}(W)$ is finite over $\mathcal{Z}(W)$
3. Study the *generic* structure of $\mathcal{D}(W)$ over $\mathcal{Z}(W)$
   Rephrased: In a neighborhood of a generic point of Spec($\mathcal{Z}(W)$), what does the algebra $\mathcal{D}(W)$ look like?

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Operator Adjoints

Theorem (Grünbaum-Tirao)

The algebra $\mathcal{D}(W)$ has a ***adjoint involution****:*

$$\dagger : \mathcal{D}(W) \to \mathcal{D}(W), \quad \mathfrak{D} \mapsto \mathfrak{D}^\dagger$$

$$\langle P(x) \cdot \mathfrak{D}, Q(x) \rangle_W = \langle P(x), Q(x) \cdot \mathfrak{D}^\dagger \rangle_W.$$

- if $W(x)$ is smooth

$$\mathfrak{D}^\dagger = W(x)\mathfrak{D}^* W(x)^{-1}$$

$$\left(\sum_k \partial_x^k A_k(x) \right)^* = \sum_k (-1)^k A_k(x) \partial_x^k.$$

New properties

1. $\mathcal{D}(W)$ is affine (finitely generated)
2. the center $\mathcal{Z}(W)$ of $\mathcal{D}(W)$ is affine
3. the ring $\mathcal{D}(W)$ is module finite over $\mathcal{Z}(W)$
4. $\mathcal{Z}(W)$ is reduced and Krull dimension 1
5. $\mathcal{D}(W)$ is a semiprime PI-algebra of GK-dim 1
6. $\mathcal{D}(W)$ is generically Azumaya over $\mathcal{Z}(W)$

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i(W)).$$

7. the previous isomorphism is involutive

Affineness

- It's not at all obvious that $\mathcal{D}(W)$ or $\mathcal{Z}(W)$ must be affine!
- $\mathcal{A} = \begin{pmatrix} \mathbb{C} & \mathbb{C}[\partial_x] \\ 0 & \mathbb{C} \end{pmatrix}$ is *commutative* and not affine

Theorem (_-Yakimov)

The algebra $\mathcal{D}(W)$ and its center $\mathcal{Z}(W)$ are both affine.

- idea: use Grünbaum's adjoint + lots of algebra

Sketch of proof

We use the eigenvalue homomorphism

$$\mathcal{D}(W) \hookrightarrow M_N(\mathbb{C}[n]), \;\; \mathfrak{D} \mapsto \Lambda(\mathfrak{D})(n)$$

$$\Lambda(\mathfrak{D})(n)P(x,n) = P(x,n) \cdot \mathfrak{D}.$$

- Existence of adjoints implies the image of $\mathcal{D}(W)$ has a diagonalizable basis
- Λ embeds $\mathcal{Z}(W)$ into a simultaneously diagonalizable subalgebra of $M_N(\mathbb{C}[n])$
- Such a subalgebra is finitely generated!

Sketch of proof

- Finiteness of $\mathcal{D}(W)$ is even harder!
- We consider an large subalgebra of $M_N(\mathbb{C}[n])$ containing the image of $\mathcal{D}(W)$
- We use Artin-Wedderburn and Tsen to prove it's an order (hence affine)
- The order is not commutative, but is a *centralizing* extension
- We apply Montgomery and Small's extension of Artin-Tate

Generic structure

Theorem (Posner)

A prime PI algebra is generically a central simple algebra over its center.

- our algebra $\mathcal{D}(W)$ is a PI algebra (embeds into a matrix ring)
- unfortunately it is not prime
- it is semiprime and Krull dimension 1

Theorem (_-Yakimov)

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i(W)).$$

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Full weights

Definition

The **module rank** of $\mathcal{D}(W)$ is $n_1 + n_2 + \cdots + n_r$ from the previous theorem. If the rank is N, we say that $W(x)$ is **full.**

Theorem (_-Yakimov)

If $W(x)$ is full, then $W(x)$ is a noncommutative bispectral Darboux transformation of a direct sum of classical weights.

$$W(x) = T(x) diag(r_1(x), r_2(x), \dots, r_n(x)) T(x)^*.$$

$$C(n)P(x,n) = diag(p_1(x,n), p_2(x,n), \dots, p_N(x,n)) \cdot \mathfrak{U}.$$

Sketch of proof

- fullness means we can choose nonzero $\mathfrak{V}_1, \ldots, \mathfrak{V}_N \in \mathcal{D}(W)$ with

$$\mathfrak{V}_i \mathfrak{V}_j = 0, \quad i \neq j.$$

- can take the $\mathfrak{V}_i$ to be W-symmetric
- define modules

$$\mathcal{M}_i = \{\vec{\mathfrak{w}} \in \Omega(x)^{\oplus N} : \vec{\mathfrak{w}}^T \mathfrak{V}_j = \vec{0}^T \ \forall j \neq i\}.$$

- $\Omega(x)$, the algebra of differential operators with rational coefficients, is a noncommutative PID:

$$\mathcal{M}_i = \Omega(x)\vec{\mathfrak{u}}_i$$

Sketch of proof

- using $\mathcal{M}_i$, define a matrix differential operator

$$\mathfrak{U} = [\vec{\mathfrak{u}_1}\ \vec{\mathfrak{u}_2}\ \dots\ \vec{\mathfrak{u}_N}]^T, \quad \vec{\mathfrak{u}_i} = \sum_{j=0}^{\ell_i} \partial_x^j \vec{u}_{ji}(x)$$

$$U(x) = [\vec{u}_{\ell_1 1}(x)\ \vec{u}_{\ell_2 2}(x)\ \dots\ \vec{u}_{\ell_N N}(x)]^T$$

- Then

$$R(x) := U(x)W(x)U(x)^* = \text{diag}(r_1(x), \dots, r_N(x)) \text{ is diagonal.}$$

$$\mathfrak{U}W(x)\mathfrak{U}^* R(x)^{-1} = \text{diag}(\mathfrak{d}_1, \mathfrak{d}_2, \dots, \mathfrak{d}_N).$$

Sketch of proof

- $p_i(x, n)$ the sequence of orthogonal polys for $r_i(x)$
- then sequence of matrix-valued functions

$$P(x, n) = \operatorname{diag}(p_1(x, n), p_2(x, n), \ldots, p_N(x, n)) \cdot \mathfrak{U}$$

- satisfies

$$P(x, n) \cdot W(x)\mathfrak{U}^* R(x)^{-1}\mathfrak{U} = \operatorname{diag}(\lambda_1(n), \ldots, \lambda_N(n))P(x, n).$$

$$\int P(x, m)W(x)P(x, n)^* dx = 0, \quad m \neq n.$$

Example

- Consider the weight matrix

$$W(x) = e^{-x^2}\begin{pmatrix} 1+a^2x^2 & ax \\ ax & 1 \end{pmatrix}.$$

- $\mathcal{D}(W)$ contains

$$\mathfrak{D}_1 = \partial_x^2 I + \partial_x \begin{pmatrix} -2x & a \\ 0 & -2x \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathfrak{D}_2 = \partial_x^2 \begin{pmatrix} -a^2/4 & a^3x/4 \\ 0 & 0 \end{pmatrix} + \partial_x \begin{pmatrix} 0 & a/2 \\ -a/2 & a^2x/2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathfrak{D}_3 = \partial_x^2 \begin{pmatrix} -a^2x/2 & a^3x^2/2 \\ -a/2 & a^2x/2 \end{pmatrix} + \partial_x \begin{pmatrix} -(a^2+1) & a(a^2+2) \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a+2/a \\ 0 & 0 \end{pmatrix}$$

$$\mathfrak{D}_4 = \partial_x^2 \begin{pmatrix} -a^3x/4 & a^2(a^2x^2-1)/4 \\ -a^2/4 & a^3x/4 \end{pmatrix} + \partial_x \begin{pmatrix} -a^3/2 & a^2(a^2+2)x/2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a^2/2+1 \\ 1 & 0 \end{pmatrix}$$

Example

- we have $\mathfrak{V}_1\mathfrak{V}_2 = 0$ for

$$\mathfrak{V}_1 = \mathfrak{D}_2, \quad \mathfrak{V}_2 = a^2\mathfrak{D}_1 + 4\mathfrak{D}_2 - 4I$$

- the modules are

$$\mathcal{M}_1 = \Omega(x)\begin{pmatrix} \partial_x a/2 \\ -\partial_x a^2 x/2 - 1 \end{pmatrix}, \quad \mathcal{M}_2 = \Omega(x)\begin{pmatrix} -1 \\ \partial_x a/2 \end{pmatrix}$$

- therefore

$$\mathfrak{U} = \begin{pmatrix} \partial_x a/2 & -\partial_x a^2 x/2 - 1 \\ -1 & \partial_x a/2 \end{pmatrix}, \quad U(x) = \begin{pmatrix} a/2 & -a^2 x/2 \\ 0 & a/2 \end{pmatrix}.$$

Example

- the weight satisfies

$$R(x) = U(x)W(x)U(x)^* = (a^2/4)e^{-x^2}I$$

- so $W(x)$ is a noncommutative bispectral Darboux transformation of the Hermite weight
- orthogonal matrix polynomials for $W(x)$ are given in terms of Hermite polynomials by

$$P(x,n) := p_{\text{herm}}(x,n)I \cdot \mathfrak{U},$$

Thanks for listening!

- New paper: https://arxiv.org/abs/1803.04405
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