Fourier Algebras

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Outline

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Time and Band-limiting

When sending a signal to a friend there are two natural limitations:

1. the range of frequencies that you can communicate, and
2. the length of time you can communicate for

Thus a basic communcation amounts to

$$\begin{aligned} T(f(t)) &= 1_{[0,\tau]}(t)\mathcal{F}^*(\overbrace{\chi_{[-\kappa,\kappa]}(k)}^{\text{limits frequencies}}(\mathcal{F}(\overbrace{1_{[0,\tau]}(t)}^{\text{limits times}} f(t))) \\ &= 1_{[0,\tau]}(t)\int_0^\tau 2K(s,t)f(s)ds, \end{aligned}$$

where $\mathcal{F}$ is the Fourier transform an $K(s,t)$ is the sinc kernel

$$K(s,t) = \frac{\sin\kappa(s-t)}{s-t}.$$

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Time and Band-limiting

Problem (Shannon 1940's)

What is the best quality of data that can be sent over a time period $[0, \tau]$ with a limited frequency range $[-\kappa, \kappa]$? ***Mathematically****, what are the eigenfunctions of the time and band-limiting operator T?*

Idea (Landau,Pollak,Slepian 1960's)

The following differential operator commutes with T

$$D(t, \partial_t) = \partial_t(\tau^2 - t^2)\partial_t - \kappa^2 t^2.$$

Eigenfunctions = solutions of differential equation!

Prolate-spheroidal Operators

Definition

An integral operator

$$T : f(x) \mapsto \int K(x,y)f(y)dy$$

which commutes with a nonconstant differential operator $D(x, \partial_x)$ *is called* ***prolate-spheroidal***.

- Tracy and Widom (1990's): other prolate-spheroidal operators coming from random matrix theory
- Duistermaat and Grunbaum: known prolate-spheroidal operators are related to **bispectral functions**

Bispectral Functions

Definition

*Let $\mathcal{S}_x \subseteq \mathcal{R}_x$ and $\mathcal{S}_z \subseteq \mathcal{R}_z$ be algebras, and $\mathcal{M}$ be a $\mathcal{R}_x, \mathcal{R}_z$-bimodule. Then $\Psi \in \mathcal{M}$ is **bispectral** if $\exists$ $L_x \in \mathcal{R}_x$, $R_z \in \mathcal{R}_z$, $F_x \in \mathcal{S}_x$, and $G_z \in \mathcal{S}_z$ sat.*

$$L_x \cdot \Psi = \Psi \cdot G_z \;\text{ and }\; \Psi \cdot R_z = F_x \cdot \Psi.$$

Classical case:

- $\mathcal{S}_x = \mathbb{C}[x]$, $\mathcal{R}_x = \mathbb{C}(x)[\partial_x]$
- $\mathcal{S}_z = \mathbb{C}[z]$, $\mathcal{R}_z = \mathbb{C}(z)[\partial_z]^{op}$
- $\mathcal{M}$ = holomorphic functions on $\mathbb{C} \times \mathbb{C}$

Examples of Bispectral Functions

- The exponential function $\psi(x,z) = e^{xz}$ is bispectral since

$$\partial_x \cdot \psi(x,z) = \psi(x,z)z \;\text{ and }\; \psi(x,z)\cdot\partial_z = x\psi(x,z).$$

- The Airy function $\psi(x,z) = \mathrm{Ai}(x+z)$ is bispectral since

$$(\partial_x^2 - x)\cdot\psi(x,z) = \psi(x,z)z \;\text{ and }\; \psi(x,z)\cdot(\partial_z^2 - z) = x\psi(x,z).$$

- The function $\psi(x,z) = \sqrt{xz}K_{\nu+1/2}(xz)$, for K_ν the modified Bessel function of the second kind, is bispectral since

$$\left(\partial_x^2 - \frac{\nu(\nu+1)}{x^2}\right)\cdot\psi(x,z) = \psi(x,z)z^2 \;\text{ and }\; \psi(x,z)\cdot\left(\partial_z^2 - \frac{\nu(\nu+1)}{z^2}\right) = x^2\psi(x,z).$$

Nonclassical Examples

$$\Psi(x,z) = e^{xz}\left[I + \begin{pmatrix} -1/xz & 1/x^2z \\ 0 & -1/xz \end{pmatrix}\right]$$

Then $\Psi(x,z)$ is bispectral since

$$\left[\partial_x^2 I + \begin{pmatrix} -2/x^2 & 4/x^3 \\ 0 & -2/x^2 \end{pmatrix}\right] \cdot \Psi(x,z) = \Psi(x,z)z^2.$$

$$\Psi(x,z) \cdot \left[\partial_z^3 I - 3\partial_z \frac{1}{z^2} I + \begin{pmatrix} 3/z^3 & 3/z^2 \\ 0 & 3/z^3 \end{pmatrix}\right] = x^3\Psi(x,z).$$

Nonclassical Examples

- Let $P(n,x)$ be a solution of the **matrix Bochner problem**: a sequence of $N \times N$ monic orthogonal matrix polynomials satisfying the matrix-valued differential equation

  $$\Lambda(n)P(n,x) = P''(n,x)A_2(x) + P'(n,x)A_1(x) + P(n,x)A_0(x)$$

  for some sequence of $N \times N$ matrices $\Lambda(0), \Lambda(1), \dots$. Then $P(n,x)$ is bispectral since it also satisfies a three-term recursion relation

  $$P(n+1,x) + B(n)P(n,x) + C(n)P(n-1,x) = P(n,x)x.$$

Outline

Bispectral Algebras

Definition

Let Ψ *be bispectral. The left and right bispectral algebras* $\mathcal{B}_x(\Psi)$ *and* $\mathcal{B}_z(\Psi)$ *are*

$$\mathcal{B}_x(\Psi) = \{L_x \in \mathcal{R}_x : \exists\, G_z \in \mathcal{S}_z,\ L_x \cdot \Psi = \Psi \cdot S_z\}.$$

$$\mathcal{B}_z(\Psi) = \{R_z \in \mathcal{R}_z : \exists\, F_x \in \mathcal{S}_x,\ F_x \cdot \Psi = \Psi \cdot R_z\}.$$

- in the classical situation, $\mathcal{B}_x(\Psi)$ is commutative
- Wilson's insight: classify bispectral functions according to their (left) bispectral algebras.

Fourier Algebras

Definition

Let Ψ be bispectral. The left and right Fourier algebras $\mathcal{F}_x(\Psi)$ and $\mathcal{F}_z(\Psi)$ are

$$\mathcal{F}_x(\Psi) = \{L_x \in \mathcal{R}_x :\ \exists\, R_z \in \mathcal{R}_z,\ L_x \cdot \Psi = \Psi \cdot R_z\}.$$

$$\mathcal{F}_z(\Psi) = \{R_z \in \mathcal{R}_z :\ \exists\, L_x \in \mathcal{R}_x,\ L_x \cdot \Psi = \Psi \cdot R_z\}.$$

- if Ψ has no left or right annihilator, the algebras $\mathcal{F}_x(\Psi)$ and $\mathcal{F}_z(\Psi)$ are isomorphic via the **generalized Fourier map**:

$$b_\Psi : L_x \mapsto R_z \text{ such that } L_x \cdot \Psi = \Psi \cdot R_z.$$

Terminology

The terminology comes from the example $\psi(x, z) = e^{xz}$.

Example

Consider the exponential bispectral function $\psi(x, z) = e^{xz}$. The left and right Fourier algebras are:

$$\mathcal{F}_x(\psi) = \mathbb{C}[x][\partial_x],$$

$$\mathcal{F}_z(\psi) = \mathbb{C}[z][\partial_z]^{op},$$

and the generalized Fourier map b_ψ is exactly the Fourier map!

$$b_\psi : \sum_{m,n} a_{mn} x^m \partial_x^n \mapsto \sum_{mn} a_{mn} \partial_z^m z^n.$$

Bispectral Darboux transformations

Take $F, \widetilde{F} \in \mathcal{S}_x$, $P, \widetilde{P} \in \mathcal{F}_x(\Psi)$ satisfying

$$\widetilde{P}\widetilde{F}^{-1}F^{-1}P \cdot \Psi = \Psi \widetilde{G}G.$$

This automatically means

$$\Psi \cdot b_\Psi(P)G^{-1}\widetilde{G}^{-1}b_\Psi(\widetilde{P}) = F\widetilde{F}\Psi.$$

The **bispectral Darboux transformation** $\widetilde{\Psi}$ is

$$\widetilde{\Psi} = F^{-1}P \cdot \Psi G^{-1}.$$

These are Darboux transformations preserving bispectrality!

$$F^{-1}P\widetilde{P}\widetilde{F}^{-1} \cdot \widetilde{\Psi} = \widetilde{\Psi}\widetilde{G}G$$

$$\widetilde{\Psi} \cdot \widetilde{G}^{-1}b_\Psi(\widetilde{P})b_\Psi(P)G^{-1} = F\widetilde{F}\widetilde{\Psi}$$

Geometric Interpretation of Ψ, $\mathcal{B}_x(\Psi)$, $\mathcal{F}_x(\Psi)$

- bispectral operators generate a spectral curve

$$\mathcal{B}_x(\Psi) \iff \text{compact Riemann surface } X$$

- eigenfunctions define a vector bundle

$$\Psi \iff \text{vector bundle } \mathcal{V} \text{ on } X$$

- the Fourier algebra is intrinsic

$$\mathcal{F}_x(\Psi) \iff \text{differential operators on } \mathcal{V}.$$

In the classical case, this is literal!

Example

Consider the bispectral function $\psi(x,z) = e^{xz}\left(1 - \frac{1}{xz}\right)$.

- In particular

$$L(x,\partial_x)\cdot\psi(x,z) = \psi(x,z)z^2 \text{ and } \widetilde{L}(x,\partial_x)\cdot\psi(x,z) = \psi(x,z)z^3.$$

$$x^2\psi(x,z) = \psi(x,z)\cdot L(z,\partial_z) \text{ and } x^3\psi(x,z) = \psi(x,z)\cdot\widetilde{L}(z,\partial_z).$$

$$L(x,\partial_x) = \partial_x^2 - \frac{2}{x^2},\ \ \widetilde{L}(x,\partial_x) = \partial_x^3 - \frac{3}{2x^2}\partial_x - \frac{3}{2x^3}.$$

- The left and right bispectral algebras are given by

$$B_x(\psi) = \mathbb{C}[L(x,\partial_x),\widetilde{L}(x,\partial_x)],\ \ B_z(\psi) = \mathbb{C}[L(z,\partial_z),\widetilde{L}(z,\partial_z)],$$

Example continued

- The Fourier algebra $\mathcal{F}_x(\psi)$ is generated by $L(x, \partial_x)$, $\widetilde{L}(x, \partial_x)$, x^2, and x^3
- Since $(z^2)^3 = (z^3)^2$, the generalize Fourier map says

$$L(x, \partial_x)^3 = \widetilde{L}(x, \partial_x)^2,$$

- $\text{spec} B_x(\psi) = \text{spec}\ \mathbb{C}[x^2, x^3]$ is a cuspidal cubic curve
- $\mathcal{F}_x(\psi)$ is isomorphic to the algebra of differential operators on this curve

$$\mathcal{F}_x(\psi) = \frac{1}{x^2}\{D(x, \partial_x) : D(x, \partial_x) \cdot \mathbb{C}[x^2, x^3] \subseteq \mathbb{C}[x^2, x^3]\}x^2.$$

Outline

Prolate Spheroidal Operators

Conjecture (Duistermaat,Grünbaum 1980's)

Let Γ_1, Γ_2 be oriented paths in $\mathbb{C}$. For sufficiently nice bispectral functions $\psi(x,z)$, the integral operator T_ψ

$$T_\psi : f(z) \mapsto \int_{\Gamma_2} K(z,w)f(w)dw, \quad K_\psi(z,w) = \int_{\Gamma_1} \psi(x,z)\psi(x,w)dx$$

is prolate-spheroidal.

Example

Consider the bispectral function $\psi(x,z) = e^{ixz}$. For $\Gamma_1 = [-\kappa,\kappa]$ and $\Gamma_2 = [0,\tau]$, the operator T_ψ is the time and band-limiting operator.

Prolate-spheroidal operators

Theorem (Casper, Yakimov 2019)

Let $\psi(x,z)$ be a self-adjoint bispectral meromorphic function of rank 1 or 2 and let Γ_1, Γ_2 be sufficiently nice paths in $\mathbb{C}$. Then there exists a nonconstant, self-adjoint operator $R(z,\partial_z) \in \mathcal{F}_z(\psi)$ commuting with T_ψ. In particular T_ψ is prolate-spheroidal.

- A similar statement holds for matrix-valued time and band-limiting operators
- Idea for proof: show that $\mathcal{F}_z(\psi)$ is large.

Size of the Fourier Algebra

Bifiltration:

$$\mathcal{F}_{z,\text{sym}}^{\ell,m}(\psi) = \{R(z,\partial_z) \in \mathcal{F}_{z,\text{sym}}(\psi) : \text{ord}(R) \le \ell, \;\; \text{ord}(b_\psi^{-1}(R)) \le m\}.$$

Theorem (Casper, Yakimov 2019)

For $\psi(x,z)$ symmetric of rank 1 or 2,

$$\dim \mathcal{F}_{z,sym}^{2\ell,2m}(\psi) \ge (\ell+1)(m+1) - const.$$

- can find $R(z,\partial_z) \in \mathcal{F}_{z,\text{sym}}(\psi)$ with

$$\int_{\Gamma_1} f(z)\cdot R(z,\partial_z)g(z)dz = \int_{\Gamma_1} g(z)\cdot R(z,\partial_z)f(z)dz$$

$$\int_{\Gamma_2} g(x)b_\psi^{-1}(R)(x,\partial_x)\cdot f(x)dx = \int_{\Gamma_2} f(x)b_\psi^{-1}(R)(x,\partial_x)\cdot g(x)dx$$

Why the Fourier algebra?

$$\begin{aligned}
T_\psi(f(z)\cdot R(z,\partial_z)) &= \int_{\Gamma_1}\left(\int_{\Gamma_2} f(w)\cdot R(w,\partial_w)\psi(x,w)\right)dw\psi(x,z)dx\\
&= \int_{\Gamma_1}\left(\int_{\Gamma_2} f(w)(\psi(x,w)\cdot R(w,\partial_w))\right)dw\psi(x,z)dx\\
&= \int_{\Gamma_2} f(w)\left(\int_{\Gamma_1} b_\psi^{-1}(R)(x,\partial_x)\cdot\psi(x,w)\psi(x,z)dx\right)dw\\
&= \int_{\Gamma_2} f(w)\left(\int_{\Gamma_1} \psi(x,w)b_\psi^{-1}(R)(x,\partial_x)\cdot\psi(x,z)dx\right)dw\\
&= \int_{\Gamma_2} f(w)\left(\int_{\Gamma_1} \psi(x,w)\cdot\psi(x,z)\cdot R(z,\partial_z)dx\right)dw\\
&= T_\psi(f(z))\cdot R(z,\partial_z).
\end{aligned}$$

Outline

Orthogonal Matrix Polynomials

Definition

*A **weight matrix** $W(x)$ on $\mathbb{R}$ is an $N \times N$ Hermitian matrix valued function which is positive-definite on an interval (a, b), identically zero elsewhere, and has finite moments.*

- defines a matrix-valued inner product:

$$\langle F, G \rangle_W = \int_{\mathbb{R}} F(x) W(x) G(x)^* dx.$$

- unique sequence of matrix-valued polynomials $P(0, x)$, $P(1, x), \dots$ with $P(n, x)$ monic of degree n and

$$\langle P(m, x), P(n, x) \rangle_W = 0I, \quad m \neq n.$$

The Matrix Bochner Problem

Problem (Matrix Bochner Problem)

Find the weight matrices $W(x)$ whose polynomials $P(n,x)$ are eigenfunctions of a second-order matrix differential operator

$$P(n,x)\cdot R(x,\partial_x) = \Lambda(n)P(n,x), \quad R(x,\partial_x) = \partial_x^2 A_2(x) + \partial_x A_1(x) + A_0(x).$$

- solved by Bochner in scalar case $N = 1$: classical orthogonal polynomials
- matrix-valued case is much harder!

Fourier Algebra

The left and right Fourier algebras are characterized by

$$\mathcal{F}_n(P) = \{\mathscr{M} \text{ matrix shift op} : \exists k > 0,\ \text{ad}_{\mathscr{L}}^{k+1}(\mathscr{M}) = 0I\}.$$

$$\mathcal{F}_x(P) = \{R(x, \partial_x) \in M_N(\mathbb{C}[x][\partial_x]) : {}^{R \text{ is } W\text{-adjointable and}}_{R^\dagger \in M_N(\mathbb{C}[x][\partial_x])}\}.$$

Here, $\mathscr{L}$ is the shift operator

$$\mathscr{L} \cdot P(n, x) = P(n+1, x) + B(n)P(n, x) + C(n)P(n-1, x)$$

defining the three-term recursion relation

$$\mathscr{L} \cdot P(n, x) = P(n, x)x.$$

Bispectral Darboux Transformations

Definition

We say that $\widetilde{W}(x)$ is a ***bispectral Darboux transformation*** *of $W(x)$ if*

$$\widetilde{P}(n,x) = C(n)^{-1}P(n,x)\cdot U(x,\partial_x)Q(x)^{-1}$$

$$P(n,x) = \widetilde{C}(n)^{-1}\widetilde{P}(n,x)\cdot \widetilde{Q}(x)^{-1}\widetilde{U}(x,\partial_x)$$

for $U(x,\partial_x), \widetilde{U}(x,\partial_x) \in \mathcal{F}_x(P)$ for some matrix-valued rational functions $Q(x), \widetilde{Q}(x), C(n), \widetilde{C}(n)$.

Note:

$$P(n,x)\cdot U(x,\partial_x)Q(x)^{-1}\widetilde{Q}(x)^{-1}\widetilde{U}(x,\partial_x) = C(n)\widetilde{C}(n)P(n,x).$$

Fourier Algebra

The right bispectral algebra $D(W)$ is

$$\mathcal{D}(W) = \{R(x, \partial_x) : \exists \Lambda(n),\ P(n, x) \cdot R(x, \partial_x) = \Lambda(n) P(x, n)\}.$$

This is not commutative, but is generically a product of matrix algebras:

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{j=1}^{r} M_{n_j}(\mathcal{F}_j(W)).$$

- $\mathcal{Z}(W)$ is the center of $\mathcal{D}(W)$
- $\mathcal{F}(W) = \bigoplus_{j=1}^{r} \mathcal{F}_j(W)$ ring of fractions of $\mathcal{Z}(W)$

If $n_1 + \cdots + n_r = N$, W is called **full.**

Matrix Bochner Problem

Theorem (Casper, Yakimov 2018)

Let $W(x)$ be an $N \times N$ weight matrix whose monic orthogonal matrix polynomials $P(n,x)$ satisfy

$$\Lambda(n)P(n,x) = P(n,x)''A_2(x) + P(n,x)'A_1(x) + P(n,x)A_0,$$

with $W(x)A_2(x)$ symmetric, positive-definite on supp(W). If $W(x)$ is full, $W(x)$ is a bispectral Darboux transformation of

$$R(x) := r_1(x) \oplus \cdots \oplus r_N(x)$$

$$W(x) = T(x)R(x)T(x)^*, \quad T(x) \in M_N(\mathbb{C}(x)).$$

$$P(n,x) = C(n)^{-1}(p_1(n,x) \oplus \cdots \oplus p_N(n,x)) \cdot U(x,\partial_x).$$

Thank you!

Papers of interest:

- "Reflective prolate-spheroidal operators and the KP/KdV equations." 2019 Proc. Natl. Acad. of Sci. USA, arXiv preprint 1909.01448
- "Integral operators, bispectrality and growth of Fourier algebras." 2019 J. Reine Angew. Math, arXiv preprint 1807.09314
- "The Matrix Bochner Problem" 2018 arXiv preprint 1803.04405